SECTION 2. Management in firms and organizations

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Investment decisions reconsidered: the case of imperfect capital markets

Abstract

This paper reconsiders the theory of investment choice. Fisher (1907, 1930) and Hirshleifer (1958) suggested to base the maximum present value criterion on general microeconomics (multi-period consumption theory). Only if investment decisions are separable from the decisions to choose consumption profiles over time, they argued whether investment calculations can be possible at all. Fisher and Hirshleifer considered them possible only if there was a perfect capital market with one single rate of interest. Hirshleifer explicitly considered market imperfections where borrowing and lending rates differed, and concluded that the maximum present value criterion did not work. In the present discussion it is shown how one, while keeping their philosophy, can extend the discussion to such imperfect markets. Maximum capital value can still be applied as a criterion, but it is different for each permutation of borrowing and lending rates over the lifetime of the investment, and separability only exists when the present values under all permutations give the same ranking.

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Introduction

Criteria for investment choice remain a rather confused area of financial economics. First, there are several criteria suggested: (i) Ranking by IRR (the expected rate of return), also the "marginal efficiency of capital" as suggested in Lord Keynes's "General Theory" in 1936, (Keynes 1936); or (ii) capital values, final or initial values, calculated assuming some calculation interest rate.

It has been recognized that ranking according to the internal rate may become problematic, as the polynomial equation defining its value may have several solutions (See, for instance, Boulding, 1934). Ranking of several different investment projects according to the internal rate criterion may further come in conflict with the maximum capital value criterion.

Whenever there is a conflict between the different criteria, it is the capital value criterion that has priority. This is because it alone can be put on a solid microeconomic basis. This basis was described most clearly by Irving Fisher (Fisher, 1907; Fisher, 1930) in terms of multi-period consumption theory. If investments are made to the end of redistributing consumption expenditures over time, and the investors in addition to investment options have access to a perfect capital market, then the capital value criterion establishes conditions for separability of the investment decision and any additional decision to borrow and lend in a perfect capital market so as to satisfy individual time preferences.

The calculation rate to use in those cases is quite obvious: it is the actual rate of interest of the assumed perfect capital market. Hirshleifer (1958) makes these issues very clear. Whenever the capital market is not perfect, as in the case of lending and borrowing rates being different, separability would no longer exist, and the capital value criterion loses its microeconomic foundation.

Despite this, there has remained a lot of confusion about the proper rate of calculation to use in the case of imperfect capital markets. Some average of lending and borrowing rates has been proposed, as has the use of some subjective "desired" rate of return (See Schneider, 1944; Sir John Hicks, 1946; or Lutz, 1945, 1951). According to Hirshleifer, "*solutions to the problem ... proposed by Boulding, Samuelson, Scitovsky, and the Lutzes are ... at least in part erroneous. Their common error lies in searching for a rule or formula which would indicate optimal investment decisions* ***independently of consumption decisions****. No such search can succeed if Fisher's analysis is sound which regards investment as not an end in itself but rather a process for distributing consumption over time*."

Puu (1964, 1967) suggested that capital value calculations could still be applied in imperfect capital markets, though there were different permutations of the lending and borrowing rates to use in the calculations, and only when one project dominated in terms of all capital values did separability and objective investment decisions exist.

This argument was presented in Sweden in 1964, and, as the original proof was so messy, only part of it was translated by Puu (1967) a few years later. Like Fisher (1907, 1930) and Hirshleifer (1958), Puu (1964, 1967), left no trace at all, so the current textbooks on financial economics appear as they did decades before, with almost nonexistent reference to Fisher's argument about investment as not an end in itself but a means of redistributing consumption expenditures over time.

To be quite precise, Puu (1964, 1967) considered investment opportunities as single fixed options, whereas Fisher (1907, 1930) and Hirshleifer (1958) had considered smooth transformation curves representing frontiers of multiple production possibilities. There is, however no difference between considering isolated options, single or multiple, or some compact set, with or without a smooth boundary, as long as just the separation theorem is at issue.

1. Definitions and notation

Consider m time periods, separated by $m+1$ time points. Period t starts with time point $t-1$, and ends with time point t. An investment is a sequence of payments x_t, accruing at time points t. Hence, a vector $\mathbf{x}=\left[x_1,...,x_m\right]$ represents an investment. It is understood that payments are made at the end of each period. As a rule, the first payments for an investment are negative (cost items) and the last are positive (revenue items). If we compare different investments, we denote them by $\mathbf{x}^i=\left[x_1^i,...,x_m^i\right]$.

In addition to different investment options $\mathbf{x}^i$, the consumer has a series of expected incomes $\mathbf{y}=\left[y_1,...,y_m\right]$, given independently of the investment activity chosen, paid at time points $t=1,...,m$. We assume that incomes are paid at the end of each period, and that the period's consumption expenditures also occur at the same point of time.

Investments provide a means of transforming the expected incomes into different consumption profiles over time $\mathbf{c}=\mathbf{y}+\mathbf{x}$, where $c_t=y_t+x_t$. Consumers value consumption profiles according to some utility function $U(\mathbf{c})$, such that they always prefer more rather than less for each period.

Remark 1. *Obviously, a consumption profile* $\mathbf{c}^2$ *is preferred to another one* $c\mathbf{c}^1$, *formally* $\mathbf{c}^2 \succ \mathbf{c}^1$, *whenever* $c_t^2 > c_t^1$, *for some* t, *whereas* $c_t^2 < c_t^1$ *for no* t. *This is the case of Paretian dominance. As* $\mathbf{y}$ *is the same in both cases, the inequalities could be written in terms of* $x_t^2 x_t^1$. *Paretian dominance seldom holds for one investment compared to another. As a rule, large negative initial costs are compensated by large later revenues, or modest intermediate revenues by very large final ones. As* $\mathbf{y}$ *is a constant in all alternatives, we can absorb it in the* $\mathbf{x}^i$ *to be compared, or suppress it altogether. We can also reduce the comparison of two investments to an evaluation of the difference investment* $\mathbf{x}^2-\mathbf{x}^1=\left[x_1^2-x_1^1,...,x_m^2-x_m^1\right]$.

In addition to choosing an investment, the consumer can also as a rule further transform the consumption profile through choosing deposits and withdrawals in a bank account. Denote such a sequence of transactions $\mathbf{z}=\left[z_1,...,z_m\right]$, assuming payments only at the ends of periods. Again there are negative elements (deposits) and positive ones (withdrawals), in this case without any special structure such as positive first and negative later. Unlike the case of investments, which represent fixed options that one can take or leave, the bank transformations are represented by a set of possible vectors $\mathbf{z}\in T$, among which the customer can choose. We call $\mathbf{z}$ a transformation vector, and T the transformation set.

Let i_{r_t} denote the applicable rate of interest during time period t. Suppose we have two different rates of interest, i_S, the saving rate, applied when the customer has a positive deposit, and i_L, the loan rate, applied when the customer owes a loan to the bank. Obviously $i_S < i_L$. For each period t, the applicable rate of interest r_t can take either of these values, $r_t \in \{S, L\}$.

We denote a sequence of applicable interest rates by $\mathbf{r}=\left[r_2,...,r_m\right]$. As payments are made at the ends of the periods, there is no payment at the beginning of the first period to which we might want to apply r_1, so we suppress this first element. (For instance, the two period consumption model only uses one interest rate.) As each period's index r_t can take two values S and L, the vector $\mathbf{r}$ denotes the whole sequence of interest rates applied. Obviously, $\mathbf{r}\in M=\{S,L\}^{m-1}$, and we will use $\mathbf{r}$ as an index to specify how different capital values are calculated. There are 2^{m-1} different interest rate sequences that can be applied.

Corresponding to an interest rate i_{r_t}, we define an interest factor $I_{r_t}=\left(1+i_{r_t}\right)$. Further we define the multi-period interest factors, from period t to period τ by $X_{\mathbf{r}}^{t,\tau}=I_{r_t}\cdot...\cdot I_{r_\tau}$. For the special case $t=\tau$, we have $X_{\mathbf{r}}^{\tau,\tau}=I_{r_\tau}$, and it is also convenient to define $X_{\mathbf{r}}^{\tau+1,\tau}=1$. Note that we specified the whole sequence of interest rates $\mathbf{r}$ for identification, though some information is redundant whenever $t>2$ or $\tau<m$.

Using the interest factors we can define the final capital values (calculated to the end of the last period) of any transformation vector **z:**

$$K_{\mathbf{r}}(\mathbf{z})=\sum_{t=1}^{t=m} z_t X_{\mathbf{r}}^{t+1,m}.$$

We can likewise define intermediate capital values of the transformation vector $\mathbf{z}$, including transactions only up to period τ:

$$K_{\mathbf{r}}^{\tau}(\mathbf{z})=\sum_{t=1}^{t=\tau} z_t X_{\mathbf{r}}^{t+1,\tau} \quad \tau=1,\ldots,m-1. \tag{1}$$

Note that these are formal combinations of any transformation vector $\mathbf{z}$ with any sequence of applied interest rates $\mathbf{r}\in M$. It may be the case that a certain $\mathbf{z}$ does *not* go with an assumed $\mathbf{r}$.

The whole point of the capital value definitions is that we want to use them to define the set of admissible $\mathbf{z}$, which we will call T, the transformation set.

In order for a $\mathbf{z}$ to be *feasible* for a given $\mathbf{r}$, we require that:

$$r_{\tau+1}=\begin{cases} S & \text{if } K_{\mathbf{r}}^{\tau}(\mathbf{z})\le 0 \\ L & \text{if } K_{\mathbf{r}}^{\tau}(\mathbf{z})>0 \end{cases} \quad \tau=1,\ldots,m-1. \tag{2}$$

The rationale for this is that $K_{\mathbf{r}}^{\tau}(\mathbf{z})$ denotes the negative of the customer's balance with the bank. The sign is reversed because positive z_t are additions to consumption and must hence be withdrawn from the bank account. So, when $K_{\mathbf{r}}^{\tau}(\mathbf{z})$ is non-positive, the saving rate is applied the following period, and when it is positive, the lending rate is applied.

A set of transformation vectors $\mathbf{z}$ which fulfils the just stated conditions is feasible for the particular interest rate vector $\mathbf{r}$. We denote the set of feasible transformation vectors $B_{\mathbf{r}}$

$$B_{\mathbf{r}}=\left\{\mathbf{z}\in R^m \left| r_{\tau+1}=\begin{cases} S & \text{if } K_{\mathbf{r}}^{\tau}(\mathbf{z})\le 0 \\ L & \text{if } K_{\mathbf{r}}^{\tau}(\mathbf{z})>0 \end{cases} \quad \tau=1,\ldots,m-1 \right.\right\}. \tag{3}$$

In this way we are sure that always the correct rate of interest is applied depending on the sign of the bank account balance.

However, in addition to the stated conditions, which constrain recursively the z_t from $t=1$ to $t=m-1$ that may be combined with a given $\mathbf{r}$, there is also a constraint which must be fulfilled by the last element z_t. It must at least clear the account, which can be expressed by the condition:

$$K_{\mathbf{r}}(\mathbf{z})=\sum_{t=1}^{t=m} z_t X_{\mathbf{r}}^{t+1,m}\le 0. \tag{4}$$

The set of transformation vectors $\mathbf{z}$, which fulfil this last condition will be denoted by $C_{\mathbf{r}}$

$$C_{\mathbf{r}}=\left\{\mathbf{z}\in R^m \left| K_{\mathbf{r}}(\mathbf{z})\le 0\right.\right\}. \tag{5}$$

Accordingly, given a sequence of interest rates $\mathbf{r}$, it must hold that:

$$\mathbf{z}\in B_{\mathbf{r}}\cap C_{\mathbf{r}}.$$

But, we can choose any $\mathbf{r}\in M$, so the full definition of the admissible transformation set is:

$$T=\bigcup_{\mathbf{r}\in M}(B_{\mathbf{r}}\cap C_{\mathbf{r}}), \tag{6}$$

where any $\mathbf{z}\in T$ is admissible.

2. Derivation of T

The transformation set $T=\bigcup_{\mathbf{r}\in M}(B_{\mathbf{r}}\cap C_{\mathbf{r}})$ is not very useful as stated, because it cannot be transformed to any simple criterion in terms of capital values. To achieve this we prove the following theorem.

Theorem 1.

$$T=\bigcup_{\mathbf{r}\in M}(B_{\mathbf{r}}\cap C_{\mathbf{r}})=\bigcap_{r\in M} C_r.$$

Proof. First we prove that $\bigcap_{r\in M} C_r\subseteq\bigcup_{\mathbf{r}\in M}(B_{\mathbf{r}}\cap C_{\mathbf{r}})$. Let $\mathbf{z}\in\bigcap_{r\in M} C_{\mathbf{r}}$. Since from definition, the feasible transformation vectors $B_{\mathbf{r}}$ together exhaust all possibilities of saving and taking loans over the periods considered, for any $\mathbf{z}$ there exist $\mathbf{r}^0\in M$ such that $\mathbf{z}\in B_{\mathbf{r}^0}$ and therefore $\mathbf{z}\in B_{\mathbf{r}^0}\cap C_{\mathbf{r}^0}$, as desired.

Now we prove that $\bigcup_{\mathbf{r}\in M}(B_{\mathbf{r}}\cap C_{\mathbf{r}})\subseteq\bigcap_{r\in M} C_r$. Let $z\in\bigcup_{\mathbf{r}\in M}(B_{\mathbf{r}}\cap C_{\mathbf{r}})$. There exists $\mathbf{r}^1\in M$ such that $\mathbf{z}\in B_{\mathbf{r}^1}\cap C_{\mathbf{r}^1}$, i.e., with $K_{\mathbf{r}^1}(z)\le 0$. For a proof by contradiction, assume there exists $\mathbf{r}^2\in M$ such that $\mathbf{z}\notin C_{\mathbf{r}^2}$, i.e., $K_{\mathbf{r}^2}(z)>0$. Then we have

$$K_{\mathbf{r}^2}(z)-K_{\mathbf{r}^1}(z)>0. \tag{1}$$

On the other hand, since $X_{\mathbf{r}}^{\tau+1,\tau}=1$ it follows that $K_{\mathbf{r}^2}(z)-K_{\mathbf{r}^1}(z)$ is equal to:

$$\sum_{\tau=1}^{m-1} z_\tau X_{\mathbf{r}^2}^{\tau+1,m} - \sum_{t=1}^{m-1} z_t X_{\mathbf{r}^1}^{t+1,m} = z_1 X_{\mathbf{r}^1}^{2,1} X_{\mathbf{r}^2}^{2,m} + \sum_{\tau=2}^{m-1} z_\tau X_{\mathbf{r}^1}^{\tau+1,\tau} X_{\mathbf{r}^2}^{\tau+1,m} - \sum_{t=1}^{m-1} z_t X_{\mathbf{r}^1}^{t+1,m}. \quad (2)$$

Since

$$\sum_{\tau=2}^{m-1} z_\tau X_{\mathbf{r}^1}^{\tau+1,\tau} X_{\mathbf{r}^2}^{\tau+1,m} = \sum_{\tau=2}^{m-1}\sum_{t=1}^{\tau} z_t X_{\mathbf{r}^1}^{t+1,\tau} X_{\mathbf{r}^2}^{\tau+1,m} - \sum_{\tau=2}^{m-1}\sum_{t=1}^{\tau-1} z_t X_{\mathbf{r}^1}^{t+1,\tau} X_{\mathbf{r}^2}^{\tau+1,m}, \quad (3)$$

substituting (3) in (2), and again using the fact that $X_{\mathbf{r}^2}^{m+1,m} = 1$, we have

$$K_{\mathbf{r}^2}(z) - K_{\mathbf{r}^1}(z)$$

equal to:

$$z_1 X_{\mathbf{r}^1}^{2,1} X_{\mathbf{r}^2}^{2,m} + \sum_{\tau=2}^{m-1}\sum_{t=1}^{\tau} z_t X_{\mathbf{r}^1}^{t+1,\tau} X_{\mathbf{r}^2}^{\tau+1,m} - \sum_{\tau=2}^{m-1}\sum_{t=1}^{\tau-1} z_t X_{\mathbf{r}^1}^{t+1,\tau+1} X_{\mathbf{r}^2}^{\tau+2,m} - \sum_{t=1}^{m-1} z_t X_{\mathbf{r}^1}^{t+1,m} X_{\mathbf{r}^2}^{m+1,m}.$$

Reorganizing and changing the summation index from $\tau-1$ to τ in the second double summand we obtain that

$$K_{\mathbf{r}^2}(z) - K_{\mathbf{r}^1}(z)$$

equals:

$$\sum_{\tau=1}^{m-1}\sum_{t=1}^{\tau} z_t X_{\mathbf{r}^1}^{t+1,\tau} X_{\mathbf{r}^2}^{\tau+1,m} - \sum_{\tau=1}^{m-2}\sum_{t=1}^{\tau} z_t X_{\mathbf{r}^1}^{t+1,\tau+1} X_{\mathbf{r}^2}^{\tau+2,m} - \sum_{t=1}^{m-1} z_t X_{\mathbf{r}^1}^{t+1,m} X_{\mathbf{r}^2}^{m+1,m} = \sum_{\tau=1}^{m-1}\sum_{t=1}^{\tau} z_t X_{\mathbf{r}^1}^{t+1,\tau} X_{\mathbf{r}^2}^{\tau+1,m} - \sum_{\tau=1}^{m-1}\sum_{t=1}^{\tau} z_t X_{\mathbf{r}^1}^{t+1,\tau+1} X_{\mathbf{r}^2}^{\tau+2,m} =$$

$$= \sum_{\tau=1}^{m-1}\left(\sum_{t=1}^{\tau} z_t X_{\mathbf{r}^1}^{t+1,\tau}\right)(I_{\mathbf{r}^2_{\tau+1}} - I_{\mathbf{r}^1_{\tau+1}}) X_{\mathbf{r}^2}^{\tau+2,m} =$$

$$= \sum_{\tau=1}^{m-1} K_{\mathbf{r}^1}^{\tau}(z)(I_{\mathbf{r}^2_{\tau+1}} - I_{\mathbf{r}^1_{\tau+1}}) X_{\mathbf{r}^2}^{\tau+2,m}.$$

Now, the interest factors $X_{\mathbf{r}^2}^{\tau+2,m} > 0$ (provided interest rates are not lower than -100 percent, which would be absurd). Therefore the sign of $K_{\mathbf{r}^2}(z) - K_{\mathbf{r}^1}(z)$ depends on the signs of the products $K_{\mathbf{r}^1}^{\tau}(\mathbf{z})\left(I_{\mathbf{r}^2_{\tau+1}} - I_{\mathbf{r}^1_{\tau+1}}\right)$ alone. However, we noted that in order that $\mathbf{z} \in B_{\mathbf{r}^1}$, we must have $r_{\tau+1}^1 = S$ whenever $K_{\mathbf{r}^1}^{\tau}(\mathbf{z}) \le 0$, and $r_{\tau+1}^1 = L$ whenever $K_{\mathbf{r}^1}^{\tau}(\mathbf{z}) > 0$. We do not know anything about $r_{\tau+1}^2$, it can equal S or L, but we know that $i_S < i_L$, so the same holds for the interest factors (which are just $1+i_S$ and $1+i_L$). Accordingly, all the products $K_{\mathbf{r}^1}^{\tau}(\mathbf{z})\left(I_{\mathbf{r}^2_{\tau+1}} - I_{\mathbf{r}^1_{\tau+1}}\right) \le 0$ for $\tau = 1,\dots,m-1$, provided $\mathbf{z} \in B_{\mathbf{r}^1}$. Hence $K_{\mathbf{r}^2}(z) - K_{\mathbf{r}^1}(z) \le 0$ yielding a contradiction with (1). This finishes the proof of the theorem.

Instead of the messy union of intersections (6), which could not be intellegiably explained in plain words, we obtained a simple intersection through Theorem 1, which can be explained in terms of simple capital value conditions. The transformation set is defined as:

$$T = \left\{\mathbf{z} \in R^n \middle| K_{\mathbf{r}}(\mathbf{z}) = \sum_{t=1}^{t=m} z_t X_{\mathbf{r}}^{t+1,m} \le 0, \forall \mathbf{r} \in \mathbf{M}\right\},$$

which means that it is the set of vectors $\mathbf{z}$ for which all capital values, calculated in all the $n = 2^{m-1}$ possible ways, by choosing the saving or the loan rent for each of the $m-1$ periods, are non-positive.

Note that, in a perfect capital market, the saving and loan rates coincide, and the conditions become identical. Then T is a linear half-space in R^n, containing the origin, and there is just one capital value condition. Otherwise T is a convex pyramid in R^n pointed at the origin.

3. The investment choice problem

In this section we propose a new perspective for dealing with the problem of investment choice. Recall that an investment was represented by a vector, and if there is a choice between two investments represented by $\mathbf{x}^1$ and $\mathbf{x}^2$, then the second is to be preferred, without regard to the income vector $\mathbf{y}$, and without regard to the form of the utility function $U(\mathbf{c}) = U(\mathbf{y}+\mathbf{x}+\mathbf{z})$, if and only if

$$K_{\mathbf{r}}\left(\mathbf{x}^2\right) > K_{\mathbf{r}}\left(\mathbf{x}^1\right) \qquad \forall\, \mathbf{r} \in M$$

holds. All the capital value functions are linear, and $\mathbf{y}$ is the same constant vector whatever the choice, so we can restrict the comparison to the composite $\mathbf{x}+\mathbf{z}$, i.e., $\mathbf{x}^2+\mathbf{z}^2$ versus $\mathbf{x}^1+\mathbf{z}^1$.

Now the investment choice problem can be restated as follows.

Theorem 1 (the investment choice problem). *Investment* $\mathbf{x}^2$ *is to be preferred to investment* $\mathbf{x}^1$ *if* *and only if for any* $\mathbf{z}^1 \in T$ *we choose along with* $\mathbf{x}^1$*, there exists at least one* $\mathbf{z}^2 \in T$*, such that the sum* $\mathbf{x}^2 + \mathbf{z}^2$ *has at least one element larger than* $\mathbf{x}^1 + \mathbf{z}^1$*, at the same time as no element is smaller.*

Proof. We will prove the condition in two steps, necessity and sufficiency. So, assume that for any $\mathbf{z}^1 \in T$ we can choose a $\mathbf{z}^2 \in T$, such that $\mathbf{x}^2 + \mathbf{z}^2$ has at least one element larger and no element smaller than $\mathbf{x}^1 + \mathbf{z}^1$. As this holds for all $\mathbf{z}^1 \in T$, it also holds for the vector of zeros, which we know belongs to T. But then $K_{\mathbf{r}}\left(\mathbf{z}^1\right) = 0$ for any $\mathbf{r} \in M$.

From the fact that $\mathbf{x}^2 + \mathbf{z}^2$ compared to $\mathbf{x}^1 + \mathbf{z}^1$ has at least one larger element and no smaller one, combined with the fact that the capital value functions are linear with positive coefficients, we obviously have

$$K_{\mathbf{r}}\left(\mathbf{x}^2 + \mathbf{z}^2 - \mathbf{x}^1 - \mathbf{z}^1\right) > 0$$

for all $\mathbf{r} \in M$. Due to the linearity and the fact that $K_{\mathbf{r}}\left(\mathbf{z}^1\right) = 0$ one deduces that

$$K_{\mathbf{r}}\left(\mathbf{x}^2\right) + K_{\mathbf{r}}\left(\mathbf{z}^2\right) > K_{\mathbf{r}}\left(\mathbf{x}^1\right).$$

But, as $\mathbf{z}^2 \in T$ then $K_{\mathbf{r}}\left(\mathbf{z}^2\right) \le 0$ and hence it follows that

$$K_{\mathbf{r}}\left(\mathbf{x}^2\right) > K_{\mathbf{r}}\left(\mathbf{x}^1\right) \qquad \forall\, \mathbf{r} \in M,$$

which finishes the proof of the necessity.

Finally we prove sufficiency. To this end assume that

$$K_{\mathbf{r}}\left(\mathbf{x}^2\right) > K_{\mathbf{r}}\left(\mathbf{x}^1\right) \qquad \forall\, \mathbf{r} \in M. \tag{1}$$

As (1) is a strong inequality, there exists a sufficiently small $\delta > 0$, such that

$$K_{\mathbf{r}}\left(\mathbf{x}^2\right) \ge K_{\mathbf{r}}\left(\mathbf{x}^1\right) + \delta \qquad \forall\, \mathbf{r} \in M \tag{2}$$

still holds.

Then start from any $\mathbf{z}^1 \in T$, and construct a $\mathbf{z}^2$, such that

$$z_i^2 = \begin{cases} x_i^1 - x_i^2 + z_i^1 & \text{if } i<m \\ x_m^1 - x_m^2 + z_m^1 + \delta & \text{if } i = m. \end{cases}$$

It is obvious that $\mathbf{x}^2 + \mathbf{z}^2$ constructed in this way has at least one element larger and no element smaller than $\mathbf{x}^1 + \mathbf{z}^1$. It only remains to prove that this $\mathbf{z}^2 \in T$. To this end $K_{\mathbf{r}}\left(\mathbf{z}^2\right) \le 0$ must be fulfilled. Note that due to the construction of $\mathbf{z}^2$, the linearity of the capital value function, and, in particular, that the last coefficient is $X_{\mathbf{r}}^{m+1,m} = 1$, we have

$$K_{\mathbf{r}}(\mathbf{z}^2) = K_{\mathbf{r}}(\mathbf{x}^1) - K_{\mathbf{r}}(\mathbf{x}^2) + K_{\mathbf{r}}(\mathbf{z}^1) + \delta \quad \forall\, \mathbf{r} \in M.$$

But $\mathbf{z}^1 \in T$, so $K_{\mathbf{r}}\left(\mathbf{z}^1\right) \le 0$ and from (2) we have that $K_{\mathbf{r}}\left(\mathbf{x}^1\right) - K_{\mathbf{r}}\left(\mathbf{x}^2\right) + \delta \le 0$. Therefore

$$K_{\mathbf{r}}\left(\mathbf{z}^2\right) \le 0 \qquad \forall\, \mathbf{r} \in M,$$

which means that $\mathbf{z}^2 \in T$. This proves sufficiency and finishes the proof of the theorem.

We have thus proved that

$$K_{\mathbf{r}}\left(\mathbf{x}^2\right) > K_{\mathbf{r}}\left(\mathbf{x}^1\right) \qquad \forall\, \mathbf{r} \in M$$

is a necessary and sufficient condition for all investors, without regard to time preferences or incomes, to choose $\mathbf{x}^2$ rather than $\mathbf{x}^1$. In other words, there is an objective investment decision for all which can be separated from the consumption decision, and it can be stated in terms of some maximum capital value principle.

In this general case we have $n = 2^{m-1}$ different capital values to calculate, and if they all result in the same ordering, separability is possible. Hirshleifer (1958) considered the case of a perfect capital market where the rates of interest for saving and for loans are equal, and claimed that investment decisions based on capital values are meaningful only in a perfect capital market. The present discourse shows that this may also be possible in an imperfect capital market where rates of interest differ.

Remark 2. *We could also easily rephrase the theorem in the following way: Defining the difference investment* $\mathbf{x} = \left(\mathbf{x}^2 - \mathbf{x}^1\right)$*, and due to the linearity of the capital value functions, all decisions could be reduced to the following: Investment* $\mathbf{x}$ *is profitable for all, irrespective of incomes and time preferences iff*

$$K_{\mathbf{r}}\left(\mathbf{x}\right) > 0, \forall\, \mathbf{r} \in M$$

and it is unprofitable for all if

$$K_{\mathbf{r}}\left(\mathbf{x}\right) < 0, \forall\, \mathbf{r} \in M.$$

As there are a lot of cases where the $n = 2^{m-1}$ *capital values yield different orderings, the investments remain incomparable. The ultimate decision depends on incomes and time preferences. This perspective is also applicable when we are not comparing different investments, but want to decide whether or not to make a single investment decision.*

Remark 3. *It should be noted that we chose to compare final values. This is, however, an arbitrary choice. We could have chosen any other time point for comparing capital values, for instance the beginning of the first period, time point* 0*. The derivations and proofs work with any such choice, because all capital values would just be divided by the same positive discount factors, which does not change the ordering. Note also that we can include an initial cost at the beginning of the first period if we wish, at time point* 0*, as we will do in the following numerical cases.*

4. Examples

We will give a graphical illustration to the applicability of the investment choice theorem by means of numerical examples for the two-period case, the simplest one that can be illustrated graphically. Then we also deal with just the interest rate for one single time period. In order to make things visible we assume an internal rate of return as high as 50 percent, i.e., by the end of the second period the investor receives one and a half times the money invested in the first.

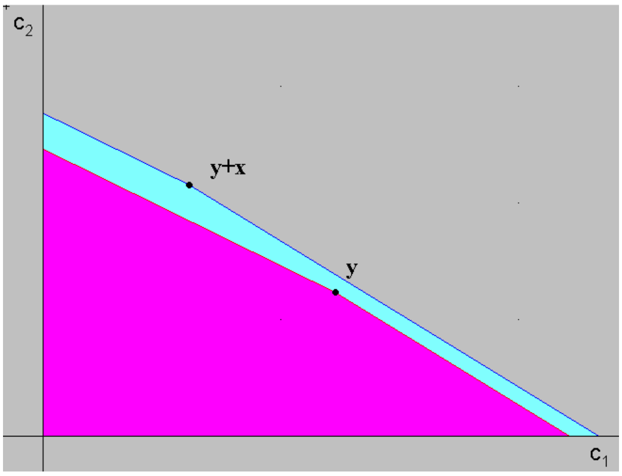

Fig. 1. Illustrative case where the investment point y + x should be chosen by everybody as its possibility area dominates that belonging to the non-investment point y. The saving rate is assumed to be 0% and the loan rate 25%. (The rates are exaggerated for increased visibility)

Possibilities for choice, consumption out of just income $\left[y_1, y_2\right]$ (right dot), or from choosing the investment $\left[y_1 + x_1, y_2 + x_2\right]$ (left dot)

The possibility set for $\left[c_1, c_2\right]$ when choosing the investment combined with a suitable transformation vector $\left[z_1, z_2\right] \in T$ dominates completely. The investment should therefore be chosen by *all*, independently of the time preferences and the location of the income vector.

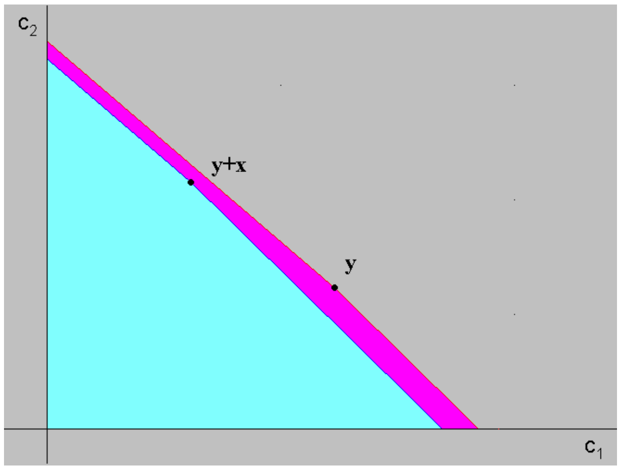

Fig. 2. Illustrative case where the non-investment point y should be chosen by everybody as its possibility area dominates that belonging to the investment y + x. The the saving rate is assumed to be 75% and the loan rate 100%. (The rates are exaggerated for increased visibility)

The possibility set for $\left[c_1, c_2\right]$ when *not* choosing the investment, combined with a suitable transformation vector $\left[z_1, z_2\right] \in T$, now dominates completely. The investment should therefore *not* be chosen by *anybody*, independently of the time preferences and the location of the income vector.

To the same end, we also assume the interest rates for bank transformations to have drastic differences as well, $r_S = 0.0$, $r_L = 0.25$ in the case on display in Figure 1, and $r_S = 0.75, r_L = 1.0$ in the case in Figure 2. It will be seen that separability exists in both cases, and objective choices for both, despite the imperfection of the capital market. In the first case the investment decision is optimal for all and in the second for none, irrespective of incomes and time preferences.

In both figures we find two dots with the same locations, $\left[y_1, y_2\right]$ (right dot) representing just incomes, and $\left[y_1 + x_1, y_2 + x_2\right]$ (left dot), representing choosing the fixed investment option. The possibility sets for consumption are

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_1 + z_1 \\ y_2 + z_2 \end{bmatrix}$$

in the first case, and

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_1 + x_1 + z_1 \\ y_2 + x_2 + z_2 \end{bmatrix}$$

in the second, where in both cases:

$$\mathbf{z} = \left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right] \in T = \left\{\mathbf{z} \in R^2 \middle| z_1(1 + r_s) + z_2 \leq 0\right\} \cap$$
$$\cap \left\{\mathbf{z} \in R^2 \middle| z_1(1 + r_L) + z_2 \leq 0\right\}.$$

As we see, in Figure 1, with low interest rates, $r_S = 0.0$, $r_L = 0.25$, the possibility set opened up when choosing the investment dominates completely, so the investment and intertemporal consumption decisions are separable, and everybody should choose the investment. This is signified by

$K_{r_S}(\mathbf{x}) > 0$ and $K_{r_L}(\mathbf{x}) > 0$

according to the general theorem above. The reader can easily verify this by using $r_S = 0.0$, $r_L = 0.25$ along with $x^2 = -1.5x^1$ (i.e., a 50 percent rate of return).

As for the case displayed in Figure 2, with high interest rates, $r_S = 0.75, r_L = 1.0$, the possibility set opened when the investment is not chosen dominates in the same way, and nobody should choose the investment. In this case

$K_{r_S}(\mathbf{x}) < 0$ and $K_{r_L}(\mathbf{x}) < 0$

Again it can be verified using $r_S = 0.75, r_L = 1.0$ along with $x^2 = -1.5x^1$.

So, we see how Hisrshleifer's (1958) results are extended to an imperfect capital market. However, assuming for instance $r_S = 0.4, r_L = 0.6$, we find that the possibility areas intersect, so the decisions are not separable, and the potential investor's time preferences and other incomes do matter.

The general point is that the Fisher-Hirshleifer argument is the only way of putting traditional capital value calculations on a firm theoretic basis. It has been shown, as in (Puu, 1964) and (Puu, 1967), that an extension of the argument is possible for imperfect capital markets, though the criteria may be restrictive, so the extension does not always work.

Conclusion

Fisher (1907, 1930) and Hirshleifer (1958) considered separability of the decisions to invest from the decisions to distribute consumption expenditures over time, which they regarded as the only reasonable rationale for the theory of investment choice. Their conclusion was rather distressing: only in the case of a perfect capital market where the rates of interest for saving and for loans are equal, and were investment decisions based on capital values meaningful. The present discussion attempts to revive some argument proposed by Puu (1964, 1967) which shows that this may also be possible in an imperfect capital market where the rates of interest differ.

The issue, raised by Fisher, may seem old, but it has been largely forgotten. In the "Web of Science" database there are but few references to Hirshleifer (1958) and none at all to Puu (1964, 1967).

In his entry in the New Palgrave, Hirshleifer (1987) repeats his former argument. The remaining impression from reading this is that the argument had no impact at all on later literature. Investors need to choose criteria, and it is uncomfortable to do without a cherished method, even if it has been shown to work only in the unrealistic case of perfect capital markets (See Schall, Sundem, Geijsbeek (1974) for an accurate survey of investment calculation methods actually used in 424 large US firms).

The argument presented here extends the Fisher/Hirshleifer analysis to imperfect capital markets where borrowing and lending rates differ, and proposes modified criteria still in terms of capital values. These modified criteria guarantee separability in imperfect markets.

Of course, these considerations apply only to cases where all future cash flows associated with investments, and all future interest rates, are foreseen with certainty. Considering uncertainty, only brings in new problematic issues, such as replacing expected probability distributions with certainty equivalents, which cannot be considered as possible choices for individual time periods, the uncertain investment project being a unique combined option. There is no need to enter these further (though essential) problems in this context.

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