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Risky strategies with payoff mean changed in 2×2 simulation-based game: a normal distribution case

Abstract

The authors investigate the Nash equilibrium payoff in the 2×2 simulation-based game where the two strategic payoffs are Normal distribution and the equilibrium payoffs are realized after the decision-making. The researchers show that the risk premiums are a part of the means of equilibrium payoffs due to the compensation of the risky strategies. The authors also show that the equilibrium payoffs are not necessary to be the same as normal distribution where it's assumed the dominant strategic payoff, but will become normal distribution only when the distance between the means of two strategic payoffs is large enough. This is revealed by the skewed and kurtosis coefficients, which approach to 0 and 3 respectively, even though they are negatively related with the average of the equilibrium payoffs. The most important result is that there is no linear relationship between the means and standard deviations of the equilibrium payoffs, but the mean of the equilibrium payoffs is a concave function with respect of the standard deviation of the equilibrium payoffs.

Keywords: decision-making, game theory, payoff uncertainty, computer simulation.
JEL Classification: C73, C63, B4.

Introduction

Most models assume probability or distribution on strategies and payoffs to represent uncertainty in game theory, unfortunately, distribution assumption is used as supplement for showing uncertainty without considering the relationship between means and variances (Varian, 2009), even without the higher-order moments. In fact, there is a risk in uncertainty when the payoffs of strategies are unsure. The risk will work on the payoffs of Nash equilibrium (NE) even when players still choose the dominant strategy (DS) because the payoffs are realized after players make their decision. Therefore, this paper builds a 2×2 game with DS and assumes that DS payoffs are normal distribution with different means and fixed variance. We can investigate how the means of the DS payoffs change the NE payoff distributions and show the relationship between the means of the DS payoffs and each moment coefficient, including the variances, skewed and kurtosis coefficients (higher-order moments) of the NE payoffs. In particular, whether the NE payoff is normal distribution it is examined by comparing with the moment coefficients of the NE payoffs and Normal distribution.

In the literature, NE that was first introduced by Nash (1950) has pure strategies and a mixed strategy which is added with probability in the view of having uncertain concept and of the same expected payoffs among strategies choosing. The effect of stochastic or uncertain concept can work on strategies or payoffs. For instance, the disturbed payoff in game is built by the mixed-strategy concept (Harsanyi, 1973). Cassidy, Field and Kirby (1972) use discrete probability and mixed strategy method to solve two-person and zero-sum game with random payoffs and provide a satisfying criterion concept. Huych, Battalio and Beil (1990) use experiment designs and strategy uncertainty in tacit coordination game and find that coordination results converge to best-efficient outcomes, moreover, coordination failures occur in the risk situation where players face strategy uncertainty and cannot play payoff-dominant outcomes. Carlsson and Damme (1993) confirm the risk dominance criterion of Hyrsanyi and Selten (1988) by payoff from a random draw in 2×2 global game with incomplete information. Yager and Alajlan (2014) present that stochastic dominance to form probability weighted means (PWM) and consider that the stochastically dominance of alternative A brings to larger PWM value. One method, simulation-based game, depends on the decision-making modeling and simulation method to solve the game outcomes. The simulation-based game considers the establishment of a game model which is closer to the reality, and the results of the game model are obtained by computer simulation. The correlated literatures are that Vorobeychik and Wellman (2008), Vorobeychik (2009) uses simulation-based game to discuss NE and the bidding strategies in auctions. Vorobeychik (2010) assumes normal distribution and uses simulation data to obtain asymptotic NE and probabilistic bounds on NE.

It is natural to expect that the uncertainty with risk has an impact on the decision-making results, and the NE payoffs are also concerned by players, although NE in game theory always pays heavily attention on the strategies that best respond to the rival's strategy. Moreover, uncertainty is always discussed by mixed-strategy and payoffs with disturbance, without complete distribution, which has the whole information of moments. Hence, we argue that a view of probability distribution

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assumption on strategic payoffs provides not only an explanation of the reality but also of the interaction between two risky strategies. We also investigate to what extent the changed means of the DS payoffs can lead to the results where the NE payoff distribution is the same as the DS payoff distribution. The differences between the literature and our paper are that (1) we assume the two strategic payoffs are normal distribution without a slight error. (2) The interaction is between two random variables, not one random variable. (3) The uncertainty is shown by the whole moments of Normal distribution, not only the mean. Therefore, the risk can be discussed in the game model.

The structure of the paper is as follows. Section 1 describes the game structure and the technique of simulation. Section 2 presents the simulated results and shows the patterns of NE coefficients. The final section concludes the paper.

1. Model and simulation procedures

1.1. The model. There are two players, Player i, i = = 1, 2, simultaneously to choose their strategies in the payoff matrix in Table 1. They have the dominant strategy, which is strategy U for Player 1 and strategy L for Player 2. The payoffs are symmetric in Table 1 with two variables, X_1 and X_2, where $X_2 > X_1$. To simplify the discussion, we only investigate the decision of Player 1. If X_1 and X_2 are certain constants, then Player 1 always chooses the dominant strategy whatever Player 2 chooses, so does Player 2. Thus, Nash equilibrium is (U, L) where Player 1 and 2 earn X_2.

Table 1. Payoff matrix of 2×2 game with DS

		Player 2	
		L	R
Player 1	U	X_2,X_2	10,X_1
	D	X_1,10	5,5

In the stochastic game, each player knows that X_1 and X_2 become random variables and are satisfied with $E(X_2) > E(X_1)$. They have the complete information of the game structure without realized values of payoffs, X_1 and X_2. The values of X_1 and X_2 are realized after players choose their strategies. They also know that X_1 and X_2 are i.i.d. Normal distribution, where X_2 is normal distribution with $1.1 \leq E(X_2) \leq 7$ and $Var(X_2) = 1$ while X_1 is standard normal distribution with $E(X_1) = 1$ and $Var(X_1) = 1$. The decision rule is $MAX(X_1, X_2)$, which is too difficult to calculate by mathematics, so that we use computer to simulate the transformation of probability distribution. We denote the NE payoff as Y, $Y = MAX(X_1, X_2)$, then simulate 60 situations where $E(X_2)$ adds 0.1 per run to form the traces of each coefficient of the NE payoff distribution. One example is that two firms compete with each other by two investing plans, respectively. Each investing plan can create uncertain profit for firms, thus, we can imply Table 1 on decision of investing plan and on knowing the benefit of investing plan in the ex ante.

1.2. Simulation procedures. We use the desktop on the Window 7 system to run C++ program, that is the probability distribution simulator. At first, we obtain a random number, labeled RND, from cumulative density function of a specific probability distribution, $F_x(x) = P(X \leq x)$. Because the values of the cumulative density function are between 0 and 1, we obtain $F_x(x) = P(X \leq x) \sim U(0, 1)$. This also represents $F_x(x) = RND$. We use the inverse function of cumulative density function to obtain the values of random variable, that is $x = F_x^{-1}(RND)$. The random variable must be the continuous-type distribution because uniform distribution is a continuous distribution. When the random variable values are considered as a data set, $\{X_1, X_2, \ldots, X_n\}$, the data size has to be increased greatly such that the discrete data set becomes continuous by the law of large numbers. Meanwhile, the frequency table can be constructed by the data set. The probability function, the graph of the distribution and corresponding coefficients can be also obtained from the frequency table. Thus, the sample frequency table is very close to the specific probability distribution, so do the coefficients of data set.

Here normal distribution, $X \sim N(\mu, \sigma^2)$, and then the algorithm equation is:

$$X_1 = \sqrt{-2\ln U_1} \times \cos(U_2 \times 2\pi)$$
$$X_2 = \sqrt{-2\ln U_1} \times \sin(U_2 \times 2\pi), \quad (1)$$

where U_1 and U_2 are i.i.d. Uniform distribution, U(0, 1) and X_1 and X_2 are i.i.d. random variables, hence, the equation will become as $X = \mu + \sigma X_1$ or $X = \mu + \sigma X_2$. Let U_1=RND1 and U_2 = RND2, run the program of:

$$X = \mu + \sqrt{-2\ln(RND1)} \times \cos(RND2 \times 2\pi) \times \sigma, \quad (2)$$

and then sort the values of X to array which have 60 million times to get frequency table. Thus, the diagram and moment coefficients of probability distribution can be calculated from the frequency table[1].

To generate the values of strategic payoffs on the condition of different parameters and then transfer to the values of the NE payoffs, the simulation steps are as follows.

[1] The distributions can be referenced at http://goo.gl/e6urC0. The software of the probability distribution simulator is authorized by C.C.C. Ltd. (http://goo.gl/okMfsY).

Step 1. Set 61 random variables follow normal distribution where X_2 is based on $1.1 \leq E(X_2) \leq 7$, with 0.1 increment, and $Var(X_2) = 1$. X_1 is based on $E(X_1) = Var(X_1) = 1$.

Step 2. Simulate and get the values from Weibull distribution, X_1 and X_2.

Step 3. Choose X_1 and X_2 with different $E(X_2)$ to do $MAX(X_1, X_2)$ and then calculate the probability distribution of the NE payoff, $Y = MAX(X_1, X_2)$.

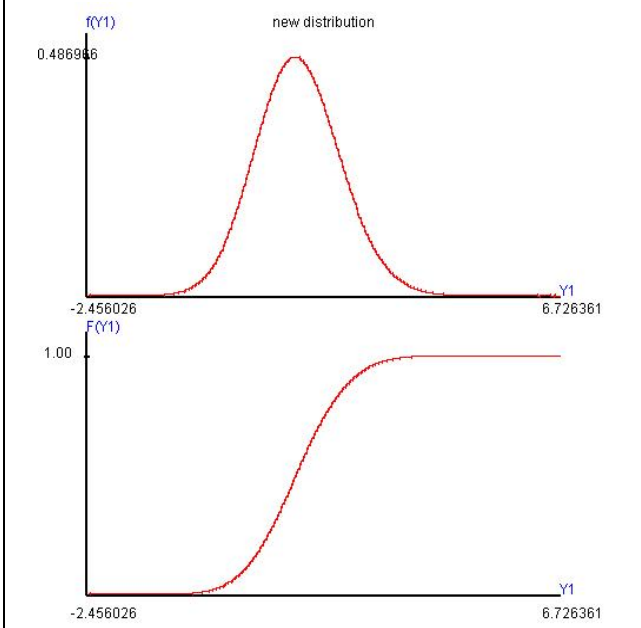
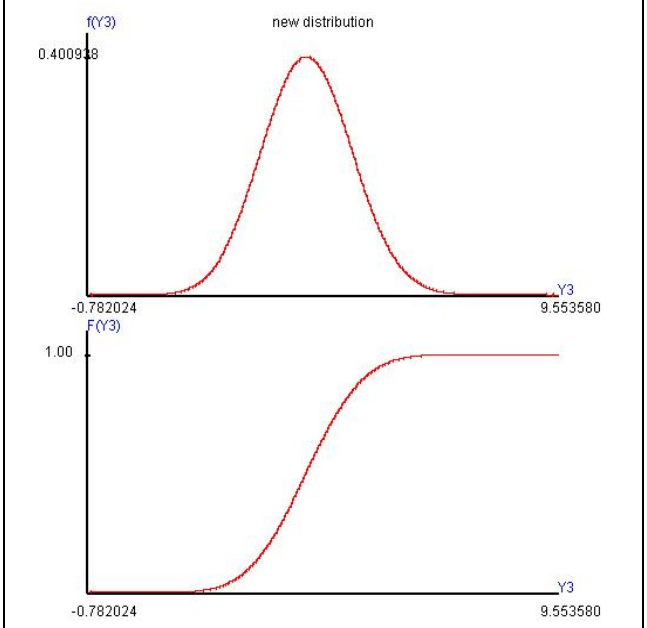
We suppose that the difference between $E(X_2) - E(X_1)$ and each moment coefficients of the NE payoff, thus, the simulation results is used to explain the moment coefficient change of the NE payoff. We denote the symbols as follows: $k = E(X_2) - E(X_1)$, Y = the NE payoff, sd(.) = the standard deviation.

2. Results

2.1. Two examples of NE payoff distribution. We simulate the model with different $E(X_2)$ given $E(X_1)$, $Var(X_1)$ and $Var(X_2)$ are fixed in the decision rule of $MAX(X_1, X_2)$. There are two examples from whole outcomes are $Y_1 = MAX(X_1 \sim N(1, 1), X_2 \sim N(1.1, 1))$ and $Y_{30} = MAX(X_1 \sim N(1, 1), X_2 \sim N(3, 1))$ in Table 2. The graphs of Y_1 and Y_{30} are not normal distribution because of the skewed and kurtosis coefficients are not 0 and 3.

Table 2 also reports that the NE payoffs become larger than the DS payoffs, that are $E(Y_1) = E(X_2) + 0.61585$ and $E(Y_{30}) = E(X_2) + 0.00847$. We find that although the NE payoffs have larger means but as $E(X_2)$ increases the difference between the means of the NE payoffs and the DS payoffs falls down from 0.61585 to 0.00847. We also find that the relationship of $Var(Y_{30}) > Var(Y_1) > Var(X_2) = 1$. Most importantly, two random payoffs in strategic decision allow that Player 1 obtains a lower risk and higher NE payoff than in the situation where only DS with payoff uncertainty. Thus, Table 2 has an evidence to reveal even Players choose a DS, the NE payoffs are not normal distribution of the DS payoff but have higher averages, lower risks, less centralized and less positive skewed.

Table 2. The shapes and coefficients of Y_1 and Y_{30}

Y_1		Y_{30}	
			
Mathematical mean	1.61585	Mathematical mean	4.00847
Variance	0.68266	Variance	0.97399
S.D.	0.82624	S.D.	0.98691
Skewed coef.	0.13782	Skewed coef.	0.05033
Kurtosis coef.	3.06339	Kurtosis coef.	2.93861

We investigate the relationship between k and coefficients of the NE payoffs as shown in Table 3. The second column shows that the higher the k is, the higher $E(Y)$ and $Var(Y)$ are, at the same time, the lower the skewed and kurtosis coefficients are. In particular, $E(Y)$ maybe almost positive and linear line with the constant linearly line. We also show that there is a 98.456% linear relationship between $E(Y)$ and $Var(Y)$. However, Table 3 may destroy the capital asset pricing model (CAPM), that shows the linear relationship between the mean and standard deviation of the portfolios because of the standard deviation is the square-root of variance, that is, $s(Y) = \sqrt{Var(Y)}$.

Table 3. The correlation between the coefficients of the NE payoff distributions

	k	Mean	Variance	Skewed coef.	Kurtosis coef.
k	1				
Mean	0.99981	1			
Variance	0.98145	0.98456	1		
Skewed coef.	-0.54296	-0.55095	-0.59471	1	
Kurtosis coef.	-0.82777	-0.82682	-0.76651	0.72119	1

To verify the distribution of the NE payoff, we have to check not only the mean and variance but also the skewed and kurtosis coefficients, which are 0 and 3, respectively. Literatures of stochastic game theory also pay less attention on skewed and kurtosis coefficients which correctly guarantee the shapes of the NE payoffs. Table 3 shows that the skewed coefficients are negative related with k, the mean and variance of the NE payoffs, but are positively related with the kurtosis coefficients. Furthermore, the increasing DS payoff leads to a relative decrease of the values of the skewed and kurtosis coefficients. Thus, the possible values of the NE payoffs are spread out and more left-skewed in the situation of high risk and high return when the DS payoffs rise up.

We also show that the row 5 shows that the NE payoff distributions become more centralized when the skewed coefficient increases. Thus, we obtain the result 1 as follows.

Result 1:

1. $\Delta E(Y) / \Delta k > 0$ and $\Delta^2 E(Y) / \Delta k^2 \rightarrow 0$.
2. $\Delta Var(Y) / \Delta k > 0$ and $\Delta^2 Var(Y) / \Delta k^2 \rightarrow 0$.
3. $\Delta E(Y) / \Delta k > 0$ and $\Delta^2 E(Y) / \Delta k^2 \rightarrow 0$.

2.2. The patterns of each coefficient. We examine how each coefficient of the NE payoff is patterned a curve when krises up. According to the top graph in Figure 1, the higher the k is, the higher $E(Y)$ is. The estimated equation is $E(Y) = 1.2616 + 0.9389k$, and shows that $E(Y)$ almost coincides with linear k, except in the interval of $k < 1$. The bottom graph of Figure 1 shows that the value of k plays very important role in the value of $Var(Y)$. At $0.1 \leq k \leq 2.3$, the variances of the NE payoff rise rapidly. The largest $Var(Y)$ is at $k = 2.3$ and then becomes increasing slowly. At $2.3 < k$, $Var(Y)$ is gradually towards 1.

By comparing with the DS payoff, Figure 1 shows that $E(Y)$ coincides with $E(X_2)$ when $k > 2$, and with $Var(X_2)$ when $k > 4$. This reveals that the DS payoff dominates that of the NE payoff when k becomes larger than a specific value. The DS payoff distribution is the same as the NE payoff distribution when $k > 4$ because of $E(Y) = E(X_2)$ and $Var(Y) = Var(X_2)$. Therefore, Players face significantly and relatively riskless NE payoff, in other words, the reduced risk is transferred to the risk premium by k, which is less than 2. When k increases, the decision-making process leads to a slight increment of $E(Y)$ from the transformation of a large decrease from $Var(X_2)$. We show the risk premium induced from the decision-making process where the NE payoff is realized after Players make their decisions in game model. We also show the mean effect on $Var(Y)$. The lower the k is, the larger the amount by which of $Var(Y)$ falls. A larger decreasing amount of $Var(Y)$ becomes a significantly slight difference between $E(Y)$ and $E(X_2)$. Thus, we obtain result 2 as follows.

Result 2:

1. $E(Y) = E(X_2)$ + risk premium
2. $\Delta(E(Y) - E(X_2)) / \Delta k < 0$
3. $\Delta Var(Y) / \Delta k > 0$ and $\Delta^2 Var(Y) / \Delta k^2 < 0$
4. $E(Y) \rightarrow E(X_2)$ and $Var(Y) \rightarrow Var(X_2)$ when $k \rightarrow 6$.

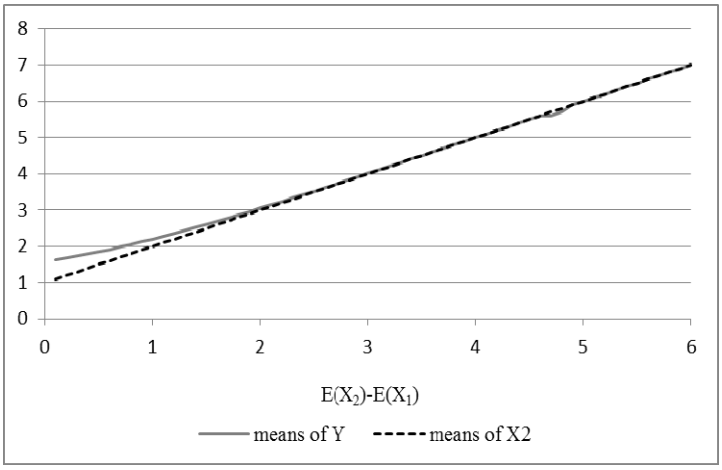


Fig. 1A. The patterns of means and variances of decision-making when *k* is from 0.1 to 6

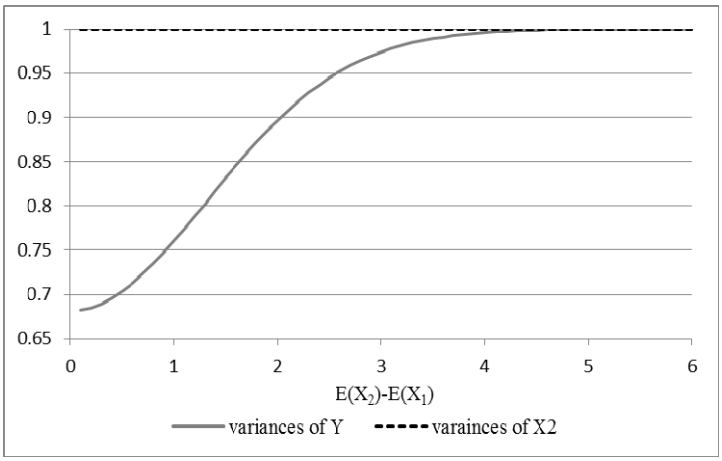


Fig. 1B. The patterns of means and variances of decision-making when *k* is from 0.1 to 6

Figure 1 seems to verify that large k leads to E(Y) = = E(X_2) and Var(Y) = E(X_2). However, to investigate whether the NE payoff distribution is the DS payoff distribution or not, we should investigate that if the skewed coefficient is 0 and the kurtosis coefficient is 3, which are correctly tested if the NE payoff is normal distribution.

Figure 2 illustrates that the patterns of skewed and kurtosis coefficients of the NE payoff when k is changed. The top graph in Figure 2 shows that the NE payoff is right-skewed for all values of k. When $0.1 \leq k \leq 1.2$, the higher the k is, the higher the skewed coefficients are. The highest skewedness occurs at $k = 1.2$ and then the skewed coefficients fall down towards 0. This implies that too small k enlarges the effect of two strategic payoff uncertainty in the decision-making process so that E(Y) is more positive-skewed than any k away from 1.2 and is also different from standard normal distribution.

The shape of skewedness also shows that small $k \leq 1.2$ leads to more seriously right-skewed NE payoff so that non-DS payoff has relatively large effect on E(Y). On the other hand, $k > 1.2$ implies that the distribution of Y is more symmetric. If $k \geq 5.1$, the skewed coefficients of the NE payoffs are obviously towards 0.

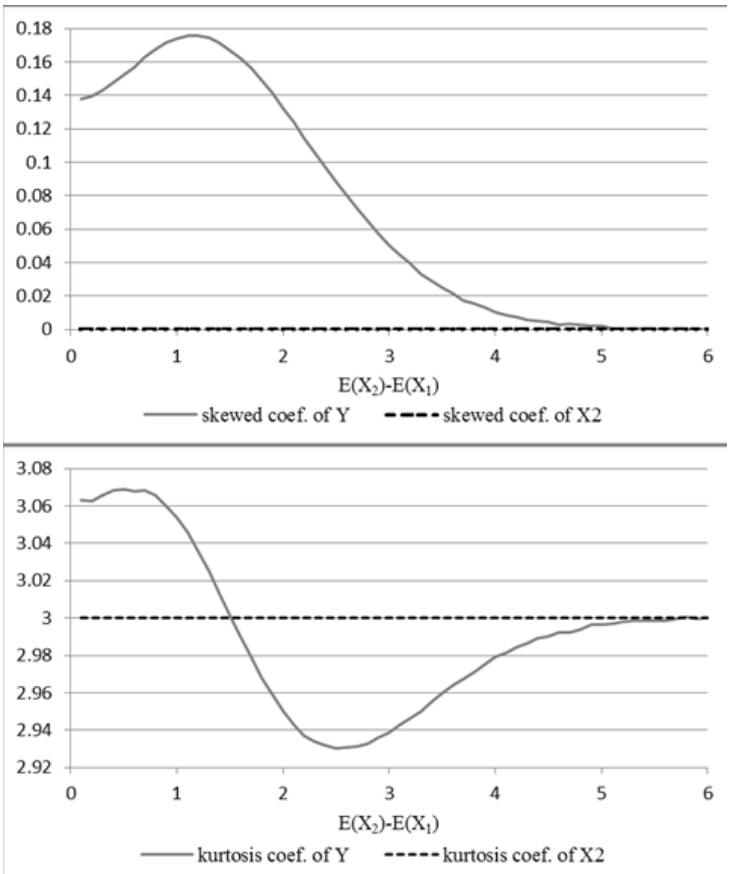


Fig.2. The decision making payoff patterns of skewed and kurtosis coefficients when k is from 0.1.to 6

The lower graph in Figure 2 illustrates the centralization of the NE payoff distributions when the k is changed from 0.1 to 6. The kurtosis coefficient is maximized at $k = 0.5$ and the minimum is at $k = 2.5$. Meanwhile, the kurtosis coefficients are larger than 3 in $0.1 \leq k \leq 1.5$ and smaller than 3 in $1.6 \leq k \leq 6$. In particular, a specific value in $1.5 < k < 1.6$ guarantees that the kurtosis coefficient is 3. This specific value is unstable point of k. Only larger k causes the kurtosis coefficients approaching to 3. By comparison with the condition of normal distribution, the specific value in $1.5 \leq k \leq 1.6$ guarantees that NE payoff distribution is not normal distribution since the skewed coefficient is not 0, at the same time, is not the payoff distribution of the dominant strategy by $\mathrm{Var}(Y) < 1$.

The NE payoff distributions are the same as normal distribution if the intersection of k from Figure 1 and 2 is larger than 4.8. This result implies that distributed payoffs cannot be viewed as any values, but the interval of k should be larger than 4.8, accurate to 0.001, and induces in the same payoff of the dominant strategy for Player 1. Thus, we obtain result 3 as follows.

Result 3:

1. The skewed coefficient is maximum at $k = 1.2$.
2. The kurtosis coefficient is maximum at $k = 0.5$ and minimum at $k = 2.5$.
3. $Y \sim$Normal if $k > 4.8$.

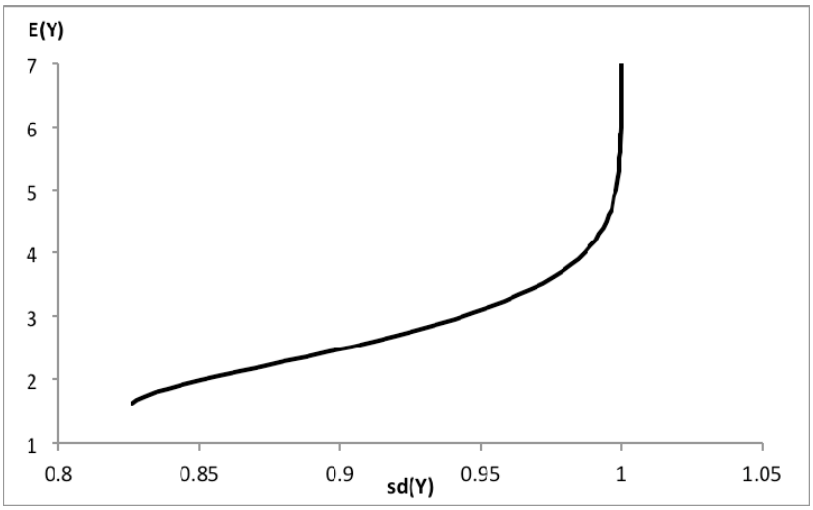


Fig. 3. The relationship between E(*Y*) and sd(*Y*)

We also examine the relationship between standard deviation and mean of the NE payoff as shown in Figure 3. Because the standard deviation is denoted as a risk, and the mean is the expected return, E(Y) is not linear related with sd(Y). The slope of E(Y) increases faster and then becomes unlimited when k approaches to 1. Figure 3 is in contrast with Varian (2009), who shows that the risky assets have linear relationship between the expected return and the risk. That is, the two random strategic payoff interact and generate nonlinear relationship between E(Y) and sd(Y) such that the higher k leads to vertical line at $\mathrm{sd}(Y) = \mathrm{sd}(X_2) = 1$ and $\mathrm{E}(Y) = \mathrm{E}(X_2)$. The major reason is the assumption of normal distributed strategic payoffs. There are two parameters of normal distribution, one is the mean effect, and the other is the risk effect. The interaction of two random variables shows that when the distance between $\mathrm{E}(X_1)$ and $\mathrm{E}(X_2)$ dominates the risk effect, the NE payoff distribution can be the same as the DS payoff distribution, which is our assumption. Thus, Players not only choose the DS, but also have the mean of the DS payoff even if the risk environment still exists.

Conclusion

We have developed a game model to study the effect of strategic payoff uncertainty on the NE payoff distribution when players make decisions before the payoffs are realized. A suitable example that can be stated in the paper is the investment portfolio which is chosen and held for at least one day. Thus, our most novel finding gives a new thinking to address (1) comparing distributions between the NE payoffs and the DS payoffs, (2) a general concept of interaction between two risky strategies, and (3) the relationship between the averages and the risks of the NE payoffs. The above issues have never beendiscussed in literature yet. We have been tested the 2×2 game model and explored its implications in a novel way. In particular, we have examined the effect of means changed on the NE payoff distribution in terms of the four moment coefficients. Undoubtedly, the equilibrium of the game includes not only the optimal strategies, but also the optimal payoffs, which include the risk premium.

In addition, we are able to shed light on the merits of these probability assumptions, such as the players'

type and mixed-strategic probability, by investigating the game model with payoff uncertainty and dominant strategies. Instead of using probability, probability distribution of strategic payoffs may have become more stringent in the treatment of the game models, but it may simply and reasonably have represented the uncertain payoff information by its higher moments. Our result shows that the means of the strategic payoffs are too close to maintain that the NE payoff distribution is the same as the DS payoff distribution. Thus, there is no reason to deny the existence of the risk on the NE payoffs when our framework can serve as building block that could usefully be added to a simulation game model with payoff uncertainty.

As with many research efforts, the present research has opened numerous paths for subsequent analysis, which we are only beginning to explore. Applying our model to the CAPM concept shows that the interaction of two risky strategies leads to a nonlinear curve of CAPM. The reason is that the larger the distance of $E(X_2) - E(X_1)$ is, the closer the NE payoff distribution becomes the DS payoff distribution. Therefore, the NE payoffs are close to Normal distribution with $sd(Y) = sd(X_2) = 1$. This result delivers that the example of the game matrix cannot set abstract values of the strategic payoffs in the situation of payoff uncertainty. The game model then can encompass learning about the decision-making process with uncertain strategies and payoffs simultaneously. Developing an integrated promotion game model along these lines of designing game structure is an interesting direction for future research.

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