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PHI-FUNCTIONS FOR ORIENTED COMPOSED 3D-OBJECTS

The paper considers a class of phi-functions for three-dimensional (3D) oriented phi-objects formed by a finite union of basic objects. We introduce truncated cones and spherical caps to extend a class of 3D basic objects, including spheres, parallelepipeds, circular cones and cylinders, as well as their complements. We construct phi-functions for extended class of basic 3D-objects and show a unified approach to derive phi-functions for oriented composed phi-objects. The Stoyan phi-function technique is employed as a powerful tool for analytical description of containment and non-overlapping constraints within the field of Packing and Cutting.

***Keywords**: 3D-objects, non-overlapping, containment, phi-functions, Packing and Cutting.*

Introduction

The placement problems in question, being a part of operational research and computational geometry, have multiple 3D applications. They can extensively be used in mechanical engineering, space engineering, car manufacturing, shipbuilding, 3D laser cutting, modeling granular media and liquids, radio-surgery treatment planning, medicine, materials science, nanotechnology, robotics, coding, pattern recognition systems, control systems, space apparatus control systems, etc.

The common placement problem lies in arranging a given set of objects within a given region (a container) in order to minimize waste of industrial materials, to minimize the use of space or maximize the number of objects, to minimize deviation from the centre of gravity, etc.

In spite of the variety of practical and scientific applications, the placement problems may be reduced to the following basic placement optimisation problem: place a set of geometric objects into a container, so that the given restrictions on the placement of the objects are fulfilled and the given objective function reaches its extreme value.

These problems are NP-hard, and, as a result, solution methodologies generally employ heuristics [e.g. 9, 12]. We refer the reader to recent tutorials [2, 19] presenting the history of placement problems and basic techniques for their solution. There are many interesting algorithms which involve 3D geometry, among them [1, 4-6, 10, 11] to mention a few. Paper [6] reviews layout algorithms for 3D objects.

The most powerful tool of mathematical modeling of placement problems as a constraint optimisation problems is the Stoyan *phi*-function technique [3, 7].

A complete class of ready-to-use radical-free *phi*-functions for all combinations of 2D basic objects considering translations, rotations and homothetic transformations of the objects is provided in the recent works [3, 7, 11].

A complete class of *phi*-functions for all combinations of oriented 3D primary objects considering non-overlapping and containment constraints is defined in [16], as well as, phi-functions for convex polytopes are introduced in [17]. Phi-functions for truncated cones one can find in [13]. A technique of constructing *phi*-functions for composed objects is given in [18].

Authors of [14, 15] define normalised phi-functions for a pair of oriented spherical caps, spherocylinders and some of basic objects.

We offer a unified way for deriving phi-functions for extended class of 3D basic objects involving spherical caps and truncated cones.

Results given in the paper based on earlier published works [3, 7, 8, 13 – 15].

Objects

In order to model mathematically real objects and their relations, we use a class of the so-called *phi*-objects (see, e.g. [7]).

Definition. Point set, $A \subset R^3$, is called a *phi*-object if it has the following characteristics: 1) *A* is canonically closed, i.e. $A = \text{cl}^*A = \text{cl}(\text{int}A)$; 2) *A* has the same homotopic type as its interior; 3) for any point $z \in \text{fr}A$ there exists an open neighbourhood U_z of *z*, such that $U_z \subset \text{int}A$ is a connected set, where int*A*, cl*A*, fr*A* are the interior, the closure and the frontier of a *phi*-object *A*.

This definition excludes point sets with isolated points, deleted points or curves, and nowhere dense point sets and objects with self-intersection of their frontiers. An important property of *phi*-objects is that if *A* is a *phi*-object, then the closure of its complement, i.e. $A^* = R^3 \setminus \text{int}A$, is a *phi*-object, too.

Each oriented object *A* may be explicitly given by its space form c (shape of *A*), metric characteristics (sizes of *A*) m and placement parameters u (a translation vector of the origin of eigen-coordinate system of *A*). Thus, a geometric information tuple $g = (c, m, u)$ generates object *A* in R^3. We suppose that *u* is a variable vector. Further we will use notation $A = A(u)$.

The class of *phi*-objects can be divided into basic and composed objects. These two groups are distinguished to facilitate the generation of *phi*-functions.

In most applications, the frontiers of 3D *phi*-objects mostly consist of flat sides, spherical, cylindrical, and conical surfaces.

Paper [16] considers a class of basic 3D *phi*-objects: a solid sphere (*S*) of radius r, a parallelepiped (*P*) given by its half-length *a*, half-width *b* and half-height *h*, a right circular cylinder (C_y) given by its radius *r* and half-height *h* and a circular cone (C_o) given by its radius *r* and height *h*. In addition, if *phi*-object *A* is a 3D primary object, then $A^* = R^3 \setminus \text{int}\,A$ is regarded as a basic object as well. In order to extend the class of basic objects we introduce a spherical cap (D) given by radius *r* of its base and its height *h* and a truncated cone (Λ) given by two radii *a*, *b* of its top and bottom bases ($a > b$ or $a < b$) and height *h*. We define spherical cap (D) in detail later. We note that in cases of centrally symmetric objects the origin of the coordinate system is always placed at the center of symmetry. We set the origin of the coordinate system of a cone, as well as a truncated cone at the center point of (bottom) base.

All other *phi*-objects that are a union of a finite number of basic objects are called composed objects.

If we let *A* be a composed *phi*-object, then

$$A = (A_1 \cup_1 A_2 \cup_2 \ldots\ldots \cup_{k-1} A_k),$$

where A_i is a basic object.

Phi-functions

Let two objects *A* and *B* be given by their geometric information tuples $g_1 = (c_1, m_1, u_1)$ and $g_2 = (c_2, m_2, u_2)$, $u_1 = (x_1, y_1, z_1)$, $u_2 = (x_2, y_2, z_2)$.

Remark. Hereinafter we suppose that at least one of these objects (either *A* or *B*) is a bounded object. It comes from the basic problem statement.

Definition. Any everywhere defined continuous function $\Phi : R^6 \to R^1$ is called a *phi*-function of 3D-objects $A(u_1)$ and $B(u_2)$, if the following characteristics hold:

$$\Phi(u_1, u_2) > 0, \text{ if } A(u_1) \cap B(u_2) = \varnothing$$

$$\Phi(u_1, u_2) = 0, \text{ if } \begin{cases} \text{fr}A(u_1) \cap \text{fr}B(u_2) \neq \varnothing \\ \text{int}\,A(u_1) \cap \text{int}\,B(u_2) = \varnothing \end{cases}.$$

$$\Phi(u_1, u_2) < 0, \text{ if } \text{int}\,A(u_1) \cap \text{int}\,B(u_2) \neq \varnothing$$

The function provides a value that indicates the state of the relations between the two objects, as described from the set-theoretical viewpoint in the previous section. Given the placement vector of each object, the *phi*-function will output a value of zero if the two objects touch; a positive value if the two objects are separated, and a negative value if the two objects intersect. We further expect that the value of the *phi*-function should give at least an estimation of the distance between the objects when they are separated and some intersection "measure" when they are overlapped.

For the sake of simplicity, we further associate *phi*-function $\Phi(u_1, u_2)$ for objects *A* and *B* with notation Φ^{AB}. For useful features of *phi*-functions we refer the reader to [3, 7].

In terms of phi-functions *non-overlapping constraint* has the form $\Phi^{AB} \geq 0$ and *containment* constraint *A⊂B*, is equivalent to $\Phi^{AB^*} \geq 0$, $B^* = R^3 \setminus \text{int}\, B$.

Phi-functions for spherical caps, truncated cones and basic 3D-objects

We offer a class of *phi*-functions for oriented 3D basic objects. We extend class of basic objects with spherical caps and truncated cones. Spherical cap D is a piece of sphere S, cutting off by plane $\{(x,y,z) : z-(r-h)=0\}$. We assume that a pole of D coincides with the center point *O* of sphere S (*O* is origin of the eigen coordinate system $OXYZ$ of *D*).

Metrical characteristics of D we define as $m_D = (r, h)$, where r is a radius of S, h is a height of *D*. We denote the base circle of *D* by C of radius $r_c = \sqrt{2rh - h^2}$ (fig. 1).

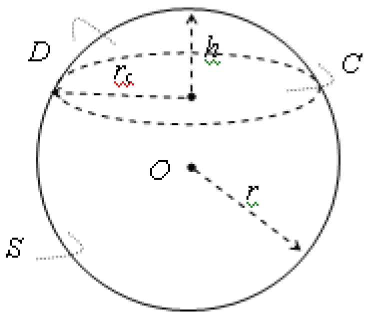


Fig. 1. Metrical characteristics of *D*

In order to derive desired phi-functions we present object *D*, taking into account its orientation (*D* and *D*'=(-1)*D*), in the form depending on metrical characteristics of *D*: a) $D = S \cap C_o$, $D' = S \cap C_o'$, if $h < r$, $O \notin D$, $O \notin D'$, $C_o' = (-1)\, C_o$. In this case, we make rigid demands to cone C_o such that the cone generator is *tangent* to *S*; b) $D = S \cap C_y$, $D' = S \cap C_y'$, if $h = r$, $O \in C_D \subset \text{fr}D$, $O \in C_D' \subset \text{fr}D'$; c) $D = S \cap \Lambda$, $D' = S \cap \Lambda'$, if $h > r$, $O \in \text{int}\, D$, $O \in \text{int}\, D'$, where

- $C_o(u_{C_o})$ is a cone of metrical characteristics $m_{C_o} = (r_{C_o}, h_{C_o})$, $r_{C_o} = r_C$, $h_{C_o} = \dfrac{h(2r-h)}{r-h}$ and $u_{C_o} = u_S + v_{C_o}$, $v_{C_o} = (0, 0, r-h)$ or $v_{C_o} = (0, 0, h-r)$ for (-1) C_o;
- $C_y(u_{C_y})$ is a cylinder of metrical characteristics $m_{C_y} = (r_{C_y}, h_{C_y})$;

  $r_{C_y} = r_C$, $h_{C_y} = h$ и $u_{C_y} = u_S + v_{C_y}$, $v_{C_y} = (0, 0, h_{C_y})$ or $v_{C_y} = (0, 0, -h_{C_y})$ for (-1) C_y;
- $\Lambda(u_\Lambda)$ is truncated cone of metrical characteristics $m_\Lambda = (a_\Lambda, b_\Lambda, h_\Lambda)$, $a_\Lambda = r_C$, $b_\Lambda = \dfrac{rh}{\sqrt{2rh - h^2}}$, $h_\Lambda = h$, and $u_\Lambda = u_S + v_\Lambda$;

  $v_\Lambda = (0, 0, r)$ or $v_\Lambda = (0, 0, -r)$ for (-1) Λ.

Let us consider a class of phi-functions for two 3D-objects A and B with placement parameters $u_1 = (x_1, y_1, z_1)$ and $u_2 = (x_2, y_2, z_2)$. One of the object is a spherical cap *D* and the other one is a basic 3D-object mentioned above. Hereinafter we assume, that $x = x_2 - x_1, y = y_2 - y_1, z = z_2 - z_1$.

Spherical cap D_1 *and sphere* S_2. Let $m_1 = (r_1, h_1)$, $m_2 = (r_2)$, then:

$$\Phi^{D_1S_2} = \max\{\Phi^{S_1S_2}, \Phi^{C_{o1}S_2}\},\ \text{if}\ h_1 < r_1,$$

$$\Phi^{D_1S_2} = \max\{\Phi^{S_1S_2}, \Phi^{C_{y1}S_2}\},\ \text{if}\ h_1 = r_1,$$

$$\Phi^{D_1S_2} = \max\{\Phi^{S_1S_2}, \Phi^{\Lambda_1S_2}\},\ \text{if}\ h_1 > r_1.$$

Spherical cap D_1 *and parallelepiped* P_2. Let $m_1 = (r_1, h_1)$, $m_2 = (a_2, b_2, h_2)$, then:

$$\Phi^{D_1P_2} = \max\{\Phi^{S_1P_2}, \Phi^{C_{o1}P_2}\},\ \text{if}\ h_1 < r_1,$$

$$\Phi^{D_1P_2} = \max\{\Phi^{S_1P_2}, \Phi^{C_{y1}P_2}\},\ \text{if}\ h_1 = r_1,$$

$$\Phi^{D_1P_2} = \max\{\Phi^{S_1P_2}, \Phi^{\Lambda_1P_2}\},\ \text{if}\ h_1 > r_1.$$

Spherical cap D_1 *and cylinder* C_{y2}. Let $m_1 = (r_1, h_1)$, $m_2 = (r_2, h_2)$, then:

$$\Phi^{D_1C_{y2}} = \max\{\Phi^{S_1C_{y2}}, \Phi^{C_{o1}C_{y2}}\},\ \text{if}\ h_1 < r_1,$$

$$\Phi^{D_1C_{y2}} = \max\{\Phi^{S_1C_{y2}}, \Phi^{C_{y1}C_{y2}}\},\ \text{if}\ h_1 = r_1,$$

$$\Phi^{D_1C_{y2}} = \max\{\Phi^{S_1C_{y2}}, \Phi^{\Lambda_1C_{y2}}\},\ \text{if}\ h_1 > r_1.$$

Spherical cap D_1 *and cone* C_{o2}. Let $m_1 = (r_1, h_1)$, $m_2 = (r_2, h_2)$, then:

$$\Phi^{D_1C_{o2}} = \max\{\Phi^{S_1C_{o2}}, \Phi^{C_{o1}C_{o2}}\},\ \text{if}\ h_1 < r_1,$$

$$\Phi^{D_1C_{o2}} = \max\{\Phi^{S_1C_{o2}}, \Phi^{C_{y1}C_{o2}}\},\ \text{if}\ h_1 = r_1,$$

$$\Phi^{D_1C_{o2}} = \max\{\Phi^{S_1C_{o2}}, \Phi^{\Lambda_1C_{o2}}\}, \text{ if } h_1 > r_1.$$

Spherical cap D_1 *and truncated cone* Λ_2. Let $m_1 = (r_1, h_1)$, $m_2 = (a_2, b_2, h_2)$, then:

$$\Phi^{D_1\Lambda_2} = \max\{\Phi^{S_1\Lambda_2}, \Phi^{C_{o1}\Lambda_2}\}, \text{ if } h_1 < r_1,$$

$$\Phi^{D_1\Lambda_2} = \max\{\Phi^{S_1\Lambda_2}, \Phi^{C_{y1}\Lambda_2}\}, \text{ if } h_1 = r_1,$$

$$\Phi^{D_1\Lambda_2} = \max\{\Phi^{S_1\Lambda_2}, \Phi^{\Lambda_1\Lambda_2}\}, \text{ if } h_1 > r_1.$$

Spherical cap D_1 *and spherical cap* D_2. Let $m_1 = (r_1, h_1)$, $m_2 = (r_2, h_2)$, then:

$$\Phi^{D_1D_2} = \max\{\Phi^{S_1S_2}, \Phi^{S_1C_{o2}}, \Phi^{C_{o1}S_2}, \Phi^{C_{o1}C_{o2}}\}$$
$$\text{if } h_1 < r_1,\ h_2 < r_2,$$

$$\Phi^{D_1D_2} = \max\{\Phi^{S_1S_2}, \Phi^{S_1C_{y2}}, \Phi^{C_{o1}S_2}, \Phi^{C_{o1}C_{y2}}\}$$
$$\text{if } h_1 < r_1,\ h_2 = r_2,$$

$$\Phi^{D_1D_2} = \max\{\Phi^{S_1S_2}, \Phi^{S_1\Lambda_2}, \Phi^{C_{o1}S_2}, \Phi^{C_{o1}\Lambda_2}\}$$
$$\text{if } h_1 < r_1,\ h_2 > r_2,$$

$$\Phi^{D_1D_2} = \max\{\Phi^{S_1S_2}, \Phi^{S_1C_{O2}}, \Phi^{C_{y1}S_2}, \Phi^{C_{y1}C_{o2}}\}$$
$$\text{if } h_1 = r_1,\ h_2 < r_2,$$

$$\Phi^{D_1D_2} = \max\{\Phi^{S_1S_2}, \Phi^{S_1C_{y2}}, \Phi^{C_{y1}S_2}, \Phi^{C_{y1}C_{y2}}\}$$
$$\text{if } h_1 = r_1,\ h_2 = r_2,$$

$$\Phi^{D_1D_2} = \max\{\Phi^{S_1S_2}, \Phi^{S_1\Lambda_2}, \Phi^{C_{y1}S_2}, \Phi^{C_{y1}\Lambda_2}\}$$
$$\text{if } h_1 = r_1,\ h_2 > r_2,$$

$$\Phi^{D_1D_2} = \max\{\Phi^{S_1S_2}, \Phi^{S_1C_{O2}}, \Phi^{\Lambda_1S_2}, \Phi^{\Lambda_1C_{o2}}\}$$
$$\text{if } h_1 > r_1,\ h_2 < r_2,$$

$$\Phi^{D_1D_2} = \max\{\Phi^{S_1S_2}, \Phi^{S_1C_{y2}}, \Phi^{\Lambda_1S_2}, \Phi^{\Lambda_1C_{y2}}\}$$
$$\text{if } h_1 > r_1,\ h_2 = r_2,$$

$$\Phi^{D_1D_2} = \max\{\Phi^{S_1S_2}, \Phi^{S_1\Lambda_2}, \Phi^{\Lambda_1S_2}, \Phi^{\Lambda_1\Lambda_2}\}$$
$$\text{if } h_1 > r_1,\ h_2 > r_2.$$

Object P_1^* *and spherical cap* D_2. Let $m_1 = (2a, 2b, 2h_1)$, $m_2 = (r_2, h_2)$, then

$$\Phi^{P^*D} = \chi,$$

where

$$\chi = \min_{i=1,\ldots,6} \chi_i,$$

$$\chi_1 = x - A,\ \chi_2 = y - B, \chi_3 = -x - A, \chi_4 = -y - B,$$

$$\chi_5 = z - (h_1 - r_2),$$

$$A = a - R,\ B = b - R,$$

$$\chi_6 = -z - (h_1 + (r_2 - h_2)),\ R = \sqrt{r_2^2 - (r_2 - h_2)^2} = r_{c_2},$$
$$\text{if } h_2 < r_2,$$

$$\chi_6 = -z - h_1,\ R = r_2, \text{ if } h_2 = r_2,$$

$$\chi_6 = -z - (h_1 - (h_2 - r_2)),\ R = r_2, \text{ if } h_2 > r_2.$$

Object $C_{y_1}^*$ *and spherical cap* D_2. Let $m_1^* = (2h_1, r_1)$, $m_2 = (r_2, h_2)$, then

$$\Phi^{C_y^*D} = \chi,$$

where

$$\chi = \min_{i=1,2,3} \chi_i,$$

$$\chi_1 = z - (h_1 - r_2),\ \chi_3 = -\sqrt{x^2 + y^2} + R,$$

$$\chi_2 = -z - (h_1 + (r_2 - h_2)),\ R = r_1 - \sqrt{r_2^2 - (r_2 - h_2)^2} =$$
$$r_1 - r_{c_2}, \text{ if } h_2 < r_2,$$

$$\chi_2 = -z - h_1,\ R = r_1 - r_2, \text{ if } h_2 = r_2,$$

$$\chi_2 = -z - (h_1 - (h_2 - r_2)),\ R = r_1 - r_2, \text{ if } h_2 > r_2.$$

Object S_1^* *and spherical cap* D_2. Let $m_1 = (r_1)$, $m_2 = (r_2, h_2)$, then:

(a) If $r_2 < r_1$, $h_2 \le r_2$ (Fig. 2):

$$\Phi^{S_1^*D_2} = \max\{\Phi^{S_1^*S_2}, \min\{\Phi^{C_1^*C_2}, f\}\}, \quad (1)$$

where

$$\Phi^{S_1^*S_2} = -(x^2 + y^2 + z^2) + (r_1 - r_2)^2,$$

$$\Phi^{C_1^*C_2} = -\sqrt{x^2 + y^2} + (r_{C_1} - r_{C_2}), \quad (2)$$

$$f = \min\{f_1, f_2\},\ f_1 = -z + \frac{(r_2 - h)r_1}{r_2},\ f_2 = z + A + B,$$

$$r_{C_1} = \sqrt{\max\{r_1^2 - (r_2 - h + z)^2, 0\}},$$

$$r_{C_2} = \sqrt{r_2^2 - (r_2 - h)^2}, \quad (3)$$

$$A = \sqrt{r_2^2 - r_{C_2}^2},\ B = \sqrt{r_1^2 - r_{C_2}^2}. \quad (4)$$

Here r_{C_1} is radius of circle C_1, generated by intersection of sphere S_1 and plane $\{(x, y, z) : z - (r_2 - h) = 0\}$, r_{C_2} is radius of circle C_2, which is the base of D_2.

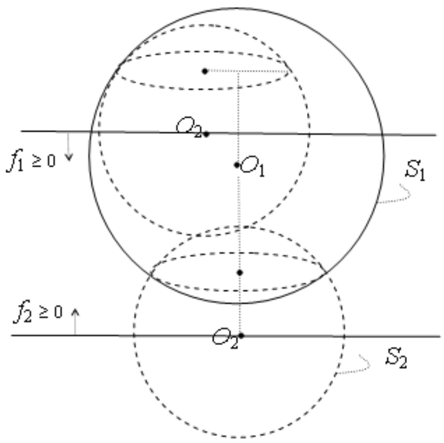


Fig. 2. Illustration for deriving $\Phi^{S_1^*D_2}$, case $r_2 < r_1$, $h_2 \le r_2$

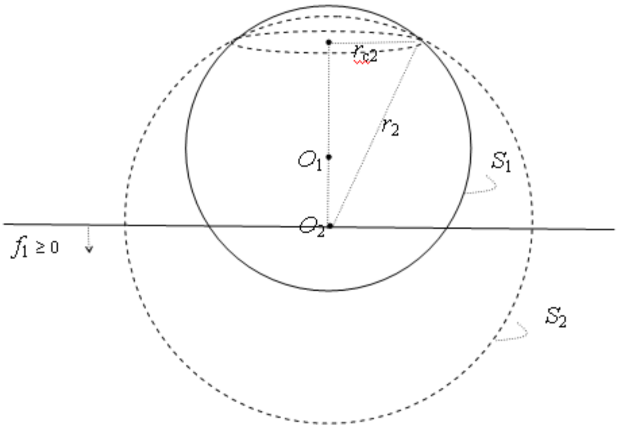
Note, that if $h_2 > r_2$, then phi-function $\Phi^{S_1^*D_2}$ takes form (1) with $f_2 = z - A + B$.

(b) If $r_2 \ge r_1$ (Fig. 3),

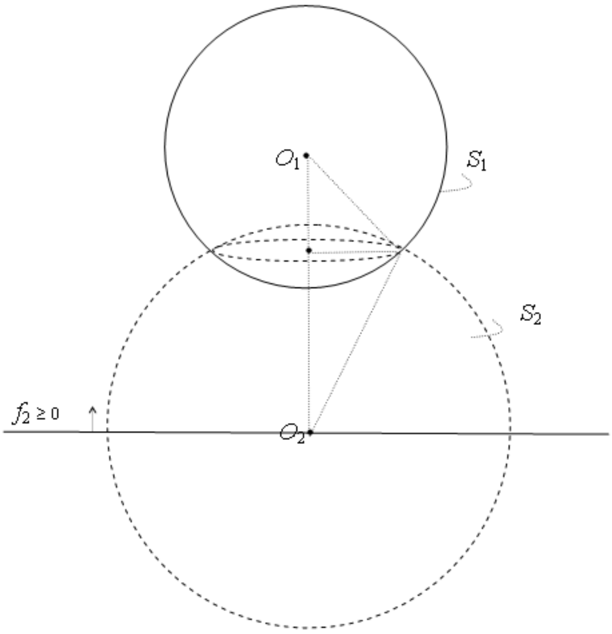
$$\Phi^{S_1^* D_2} = \min\{\Phi^{C_1^* C_2}, f\},$$

$$f_1 = -z + B - A,\ f_2 = z + B + A,$$

where $\Phi^{C_1^* C_2}$ is defined by (2), r_{C_1}, r_{C_2} and A, B are defined by (3) and (4), respectively.



a



b

Fig. 3. Illustration for deriving $\Phi^{S_1^* D_2}$, case $r_2 \ge r_1$:
a – definition of f_1, b – definition of f_2

In order to derive phi-functions mentioned above we use the following basic phi-functions.

Two truncated cones Λ_1 *and* Λ_2. Let $m_1 = (a_1, b_1, h_1)$, $m_2 = (a_2, b_2, h_2)$:

1) If $a_1 < b_1$ и $a_2 < b_2$, then

$$\Phi^{\Lambda_1 \Lambda_2} = \psi,$$

where

$$\psi = \max\{\chi, \phi\},$$

$$\chi = \max\{\chi_1, \chi_2\},\ \phi = \max\{\phi_1, \phi_2\},$$

$$\chi_1 = z - h_1,\ \chi_2 = -z - h_2$$

$$\phi_i = \frac{h_i}{l_i} \cdot \left(\sqrt{x^2 + y^2} + (-1)^{i+1} \cdot \frac{z r_i}{h_i} - r^*\right)$$

$$r^* = b_1 + b_2,\ r_i = b_i - a_i,\ l_i = \sqrt{h_i^2 + r_i^2},\ i = 1, 2.$$

2) if $a_1 < b_1$ и $a_2 > b_2$, and $\frac{r_1}{h_1} \ne \frac{r_2}{h_2}$, then

$$\Phi^{\Lambda_1 \Lambda_2} = \psi,$$

where

$$\psi = \max\{\chi, \phi\},$$

$$\chi = \max\{\chi_1, \chi_2\},\ \phi = \max\{\phi_1, \phi_2\},$$

$$\chi_1 = z - h_1,\ \chi_2 = -z - h_2,$$

$$\phi_i = \frac{h_i}{l_i}\left(\sqrt{x^2 + y^2} + \frac{z r_i}{h_i} - r_i^*\right),$$

$$r_1^* = b_1 + b_2,\ r_2^* = b_1 + a_2,\ r_1 = b_1 - a_1,\ r_2 = a_2 - b_2,$$

$$l_i = \sqrt{h_i^2 + r_i^2},\ i = 1, 2.$$

3) if $a_1 < b_1$ и $a_2 > b_2$, and $\frac{r_1}{h_1} = \frac{r_2}{h_2}$, then

$$\Phi^{\Lambda_1 \Lambda_2} = \psi,$$

where

$$\psi = \max\{\chi, \phi\},\ \chi = \max\{\chi_1, \chi_2\},$$

$$\chi_1 = z - h_1,\ \chi_2 = -z - h_2,$$

$$\phi = \frac{h_1}{l_1}\left(\sqrt{x^2 + y^2} + \frac{z r_1}{h_1} - r^*\right),$$

$$r^* = b_1 + a_2,\ r_1 = b_1 - a_1,\ l_1 = \sqrt{h_1^2 + r_1^2}.$$

Truncated cone Λ_1 *and cylinder* C_{y2}. Let $m_1 = (a_1, b_1, h_1)$, $m_2 = (r_2, h_2)$, then

$$\Phi^{\Lambda C_y} = \omega,$$

where

$$\omega = \max\{\chi, \phi\},\ \chi = \max_{i=1,2} \chi_i,\ \phi = \max_{i=1,2} \phi_i,$$

$$\chi_1 = -z - (h_1 + h_2),\ \chi_2 = z - h_2,$$

$$\phi_1 = \sqrt{x^2 + y^2} - r,\ \ r = b_1 + r_2,$$

$$\phi_2 = \alpha \cdot \left(\sqrt{x^2 + y^2} + \frac{z(b_1 - a_1)}{h_1} - r\right),$$

$$\alpha = \frac{h_1}{\sqrt{h_1^2 + (b_1 - a_1)^2}}.$$

Truncated cone Λ_1 *and cone* C_{o2}. Let $m_1 = (a_1, b_1, h_1)$, $m_2 = (b_2, h_2)$, then

$$\Phi^{\Lambda C_0} = \psi,$$

where

$$\psi = \max\{\chi, \phi\},\ \chi = \max\{\chi_1, \chi_2\},\ \phi = \max\{\phi_1, \phi_2\},$$

$$\chi_1 = z - h_1,\ \chi_2 = -z - h_2,$$

$$\phi_1 = \frac{h_1}{l_1}\cdot(\sqrt{x^2+y^2} + \frac{zr_1}{h_1} - r),$$

$$r = b_1 + b_2,\ r_1 = b_1 - a_1,\ \ l_1 = \sqrt{h_1^2 + r_1^2},$$

$$\phi_2 = \frac{h_2}{l_2}\cdot(\sqrt{x^2+y^2} - \frac{zb_2}{h_2} - r),\ l_2 = \sqrt{h_2^2 + b_2^2}.$$

Truncated cone Λ_1 *and sphere* S_2. Let $m_1 = (a_1, b_1, h_1)$, $m_2 = (r_2)$, then

$$\Phi^{\Lambda S} = \varpi,$$

where

$$\varpi = \max\{\phi, \chi_1, \chi_2, \zeta\},$$

$$\zeta_i = \min\{\varphi_i, \omega_i\},\ \zeta = \max_{i=1,2} \zeta_i,$$

$$\phi = \frac{h_1}{l_1}\cdot(\sqrt{x^2+y^2} + \frac{zr_1}{h_1} - r^*),\ h^* = h_1(1 + \frac{a_1}{r_1}) + \frac{l_1 r_2}{r_1},$$

$$r^* = \frac{r_1 h^*}{h_1},$$

$$\varphi_1 = \frac{l_1 - r_1}{h_1}\cdot\sqrt{x^2+y^2} + z - h',\ h' = h_1 + r_2 + \frac{l_1 - r_1}{h_1}\cdot a_1,$$

$$\varphi_2 = \frac{l_1 + r_1}{h_1}\cdot\sqrt{x^2+y^2} - z - h'',\ h'' = \frac{l_1 + r_1}{h_1}\cdot b_1 + r_2,$$

$$l_1 = \sqrt{h_1^2 + r_1^2},$$

$$\omega_1 = \sqrt{(f - a_1)^2 + (z - h_1)^2} - r_2,$$

$$\omega_2 = \sqrt{(f - b_1)^2 + z^2} - r_2,$$

$$\chi_2 = -z - r_2,\ \chi_1 = z - (h_1 + r_2).$$

Object S_1^* *and truncated cone* Λ_2. Let $m_1^* = (r_1)$, $m_2 = (a_2, b_2, h_2)$, then

$$\Phi^{S_1^*\Lambda_2} = \varphi,$$

where

$$\varphi = \min\{\varphi_1, \varphi_2\},$$

$$\varphi_1 = r_1 - \sqrt{(\sqrt{x^2+y^2} + a_2)^2 + (z + h_2)^2},$$

$$\varphi_2 = r_1 - \sqrt{(\sqrt{x^2+y^2} + b_2)^2 + z^2}.$$

Object P_1^* *and truncated cone* Λ_2. Let $m_1 = (2a_1, 2b_1, 2h_1)$, $m_2 = (a_2, b_2, h_2)$, then

$$\Phi^{P_1^*\Lambda_2} = \chi,$$

where

$$\chi = \min_{i=1,2..6} \chi_i,$$

$$\chi_1 = -x + (a_1 - b_2), \chi_2 = x + (a_1 - b_2),$$

$$\chi_3 = -y + (b_1 - b_2), \chi_4 = y + (b_1 - b_2),$$

$$\chi_5 = z - (h_1 - h_2), \chi_6 = -z - h_1.$$

Object C_y^* *and truncated cone* Λ_2. Let $m_1^* = (r_1, h_1)$, $m_2 = (a_2, b_2, h_2)$, then

$$\Phi^{C_y^*\Lambda_2} = \omega,$$

where

$$\omega = \min\{\chi_1, \chi_2, \varphi\},$$

$$\chi_1 = z - (h_1 - h_2),\ \ \chi_2 = -z - h_1,$$

$$\varphi = -\sqrt{x^2+y^2} + (r_1 - b_2).$$

The rest part of phi-functions for 3D basic objects one can find in [7, 8, 16].

Phi-functions for composed objects

Phi-function for composed objects operates with *phi*-functions of 3D basic objects.

Let objects *A* and *B* be given in the form

$$A = A_1 \cup \ldots \cup A_{n'},\ B = B_1 \cup \ldots \cup B_{n''},$$

where A_i and B_j are basic objects. Then *phi*-function for *A* and *B* has the form

$$\Phi^{AB} = \min\{\Phi_{ij}, i = 1, 2, ..., n', j = 1, 2, ..., n''\},$$

where Φ_{ij} is a *phi*-function for basic objects A_i and B_j.

The technique of constructing *phi*-functions for complex objects for 2D case is given in details in [3, 7, 18].

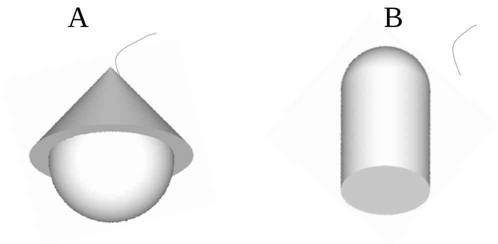


Fig. 4. Two composed 3D-objects *A* and *B*

Let us consider two composed objects *A* and *B* shown in figure 5, where

$$A = S \cup C_o \text{ and } B = D \cup C_y.$$

Thus, $\Phi^{AB} = \min\{\Phi^{SD}, \Phi^{SC_y}, \Phi^{DC_o}, \Phi^{C_yC_o}\}$,

where Φ^{SD}, Φ^{SC_y}, $\Phi^{DC_o}, \Phi^{C_yC_o}$ are basic phi-functions.

Conclusions

The concepts of the *phi*-object and the *phi*-function have their roots in topology, but *phi*-functions are very convenient for practical solution of placement problems. In this paper we derive phi-functions for extended class of basic 3D-objects, as well as a class of composed phi-objects in order to improve the performance of 3D cutting and packing algorithms. Our phi-functions can be applied for an analytical description of non-overlapping and containment constraints. In addition, the phi-functions are defined by simple formulas, which allow us to use optimisation algorithms of mathematical programming. Phi-functions allow us to enlarge the class of optimisation placement 3D-problems that can be effectively solved.

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PHI-ФУНКЦИИ ДЛЯ ОРИЕНТИРОВАННЫХ СОСТАВНЫХ 3D-ОБЪЕКТОВ

Т. Романова, Б. Рублев, А. Башук, В. Синявин

Рассматривается класс phi-функций для трехмерных (3D) ориентированных phi-объектов, которые формируются объединением конечного числа базовых объектов. С целью расширения класса базовых 3D-объектов, включающий шары, параллелепипеды, круговые цилиндры и конусы, а также их дополнения, вводятся усеченные конусы и шаровые сегменты. Приводятся phi-функции для расширенного класса базовых объектов, и предлагается единый подход к построению phi-функций для ориентированных составных 3D-объектов. Метод phi-функций Стояна используется для аналитического описания отношений непересечения и включения геометрических объектов – как эффективное средство математического моделирования задач Упаковки и Раскроя.

***Ключевые слова**: 3D-объекты, непересечение, включение, phi-функции, Упаковка и Раскрой.*

PHI-ФУНКЦІЇ ДЛЯ ОРІЄНТОВАНИХ СКЛАДЕНИХ 3D-ОБ'ЄКТІВ

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Розглядається клас phi-функцій для тривимірних (3D) орієнтованих phi-об'єктів, які формуються об'єднанням скінченної кількості базових об'єктів. До класу базових об'єктів, що включає кулі, паралелепіпеди, кругові циліндри і конуси, а також їх доповнення, вводяться усічені конуси та кульові сегменти. Наводяться phi-функції для розширеного класу базових об'єктів і пропонується єдиний підхід до побудови phi-функцій для орієнтованих складених 3D-об'єктів. Метод phi-функцій Стояна використовується для аналітичного опису відношень неперетину та включення геометричних об'єктів – як ефективний засіб математичного моделювання задач Пакування та Розкрою.

***Ключові слова:** 3D-об'єкті, неперетин, включення, phi-функції, Пакування та Розкрій.*