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Yu.I. Dorofieiev

SUBSYSTEM STABILIZATION APPROACH TO ROBUST DECENTRALIZED INVENTORY CONTROL IN SUPPLY NETWORKS

Abstract. A problem of the robust decentralized inventory control strategy synthesis for supply networks under uncertain external demand and transport time-delays and with presence of asymmetric structural constraints on states and controls is considered. Decentralized control is designed in the form of linear non-stationary feedback with respect to deviation of the current stock level from the chosen safety level and is based on solving convex optimization problems of subsystems dimension. Solvability conditions of the synthesis problem are stated in the form of linear matrix inequalities and reduced to solving semidefinite programming and one-dimensional convex optimization problems. To analyze the stability of the controlled supply network with decentralized controllers the comparison method and mathematical tool of vector Lyapunov functions is used.

Keywods: supply network, inventory control, invariant ellipsoids method, linear matrix inequality, semidefinite programming, comparison method, vector Lyapunov function.

Introduction

A supply network is a complex system consisting of interconnected agents which is engaged in the extraction of raw materials, production, storage, transportation and distribution of products to satisfy consumer demand [1]. Supply network may be represented as a directed graph with vertices corresponding to the network nodes which define types and volumes of controllable inventory and arcs which are controllable and uncontrollable flows in the network. Controllable flows describe processes of resource reprocessing and redistributing between network nodes and external supply processes for raw materials. Uncontrollable flows describe the resource demand formed by external consumers.

Operation of production links associated with supply network nodes and influence of demand from external consumers make resource stock in the network nodes change with time. This leads to the need in control methods for supply networks to construct optimal inventory control strategies, with various uncertainty factors taken into account. We take a supply network control strategy as a structure of rules of determining instants and volumes of the resupply order.

In terms of supply network control, it is reasonable to consider volumes of demand on the resources that are received at the network nodes from the external environment as external disturbances. The choice of inventory control model is defined by nature of demand. At present, the inventory control strategy with the given demand model is synthesized using the Model Predictive Control [2].

In practice, generally, there is no information for constructing a proper model of external demand needed to construct the predictive control. One of approaches to the solution of the inventory control problem under demand uncertainty is use of the concept of "unknown, but bounded" inputs [3]. The respective demand model is characterized by the interval uncertainty.

Analysis of recent research and publications

Most of the procedures for the analysis and synthesis of automatic control systems in recent decades were developed using a centralized approach, where all the information about the current system state is transferred to a single regulator, which formed the control actions for all system nodes. A lot of results on the stability and robustness of centralized control algorithms, recently obtained [4].

However, a centralized approach to the construction of the control system is characterized by significant computational complexity and the need a centralized system for collecting information. Therefore, for the control problems of supply networks the decentralized approach is perspective, in which the original optimization problem is replaced by a set of local problems of smaller dimension that can be solved in parallel and independently. At once, it is necessary to ensure the robust stability of the whole system, taking into account the availability of relationships.

Main attention focuses on the problem of robustness of a decentralized control structure [5, 6]. Synthesis of stabilizing control algorithms in the form of a static output feedback was performed with the help of Lyapunov's function using the estimate of the upper bound of non-linear terms or relationships between local subsystems using the estimate of the upper bound of non-linear terms or relationships between local subsystemsis. To get the results that are acceptable from the point of view of computational complexity, the stability conditions are formulated using the technique of Linear Matrix Inequalities (LMI) [7]. However, in this approach, decentralized control is found by solving the optimization problem whose dimension is determined by the dimension of the complete system.

To reduce the dimension of decentralized control problem for large-scale systems synthesis the concept of a diagonal or block-diagonal dominance can be applied. Using of this concept, in [8] an approach to the synthesis of decentralized control is proposed, which is based on the method of equivalent subsystems. Initially, a suitable method of synthesis decentralized controllers has been developed as a technique in the frequency domain on the basis of the Nyquist method. In this work an approach similar to the method of equivalent subsystems used for the synthesis of decentralized control in the state space. The main advantage of these approaches is that the static output feedback, providing robust stability and given performance values, is constructed for the individual subsystems, which reduces the dimension of the problem to the dimension of the subsystems. In this case, the estimates of the degree of subsystems stability obtained as a result of the optimization problem solution are considered as constraints on the level of the relationships between the subsystems. However, in this approach in the model does not take into account the structural constraints, as well as external disturbances, while they are highly significant for supply networks control.

A characteristic feature of the inventory control problem is the presence of transport time-delays caused by delays in replenishment about the moment of ordering. Also it should be noted that in the works devoted to the problem of suppression of bounded external disturbances, LMI technique is usually applied to suppress disturbances that are limited in some norm. While the specifics of the inventory control problem is a non-negative values of variables that leads to the presence of asymmetric constraints on values of the states and control actions.

The aim of this work is the synthesis of robust decentralized inventory control strategy for supply network under the action of an unknown, but bounded external demand and transport delays with the defined structural constraints on the states and control actions.

Problem formulation

Consider the supply network S, consisting of interconnected nodes $S_i, i=\overline{1,N}$, each of which is multinomenclature system, described by the discrete state-space model. As the state variables available inventory levels of resources are considered. Control actions are orders volumes for the resources supply, which are formed by nodes in the current period, as well as external disturbances is the demand orders, which arrive at the network nodes from the outside.

System behavior is determined by the equations describing the change in the stock levels of each node S_i. It is assumed that the supply network structure is known, and the states are available to direct measurement. It is also assumed that the measured values of the local states come only on their local controllers.

Transport time-delays are described using the discrete delay model. Values of delays that give the time duration of transportation and resource reprocessing at the network nodes are supposed to be known and are multiples of the sampling period. Then each node is described by a difference equation with delay

$$\mathbf{x}_i(k+1)=\mathbf{x}_i(k)+\sum_{t=0}^{\Lambda_i^{\max}}\mathbf{B}_i^t\mathbf{u}_i(k-t)+\mathbf{E}_i\mathbf{w}_i(k), \quad i=\overline{1,N}, \tag{1}$$

where $k=0,1,2,...$ is number of discrete time interval; $\mathbf{x}_i(k)\in\mathbf{R}^{n_i}$ is state vector of node S_i; $\mathbf{u}_i(k)\in\mathbf{R}^{m_i}$ is control actions vector; $\mathbf{w}_i(k)\in\mathbf{R}^{n_i}$ is external disturbances vector; $\Lambda_i^{\max}$ is discrete variable, multiple of the sampling period, that determines the maximum value of the time-delays of controlled flows between the node S_i and the network nodes that are resource suppliers for him; $\mathbf{B}_i^t\in\mathbf{R}^{n_i\times m_i}$, $t=\overline{0,\Lambda_i^{\max}}$ are control influence matrices, $\mathbf{E}_i\in\mathbf{R}^{n_i\times n_i}$ is disturbances influence matrix. Obviously, the network structure is determined by the matrices $\mathbf{B}_i^t$, $\mathbf{E}_i$, which are constructed by the methods, described in **[9]**. The vector $\mathbf{x}(k)=\left[\mathbf{x}_1^{\mathrm{T}}(k),\mathbf{x}_2^{\mathrm{T}}(k),...,\mathbf{x}_N^{\mathrm{T}}(k)\right]^{\mathrm{T}}$, which composed of the individual nodes state vectors, is the state vector of the whole system S and has dimen-

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sion $n=\sum_{i=1}^{N} n_i$.

External actions for each node S_i include the functions of external demand generated outside the network, and internal demand generated by nodes for which the node S_i is a resources supplier:

$$\mathbf{w}_i(k)=\sum_{j=1,\, j\neq i}^{N} \mathbf{\Pi}_{ij}\mathbf{u}_j(k)+\mathbf{\Pi}_i\mathbf{d}(k),$$

where $\mathbf{d}(k)\in \mathbf{R}^q$ is external demand vector; $\mathbf{\Pi}_{ij}\in \mathbf{R}^{n_i\times n_j}$, $i,j=\overline{1,N}$ are technological matrices, which are formed on the basis of the process description being implemented by the supply network: the element value $\mathbf{\Pi}_{ij}(s,t)$ is equal to the amount of resource units $s=\overline{1,n_i}$ of node S_i required to produce one resource unit $t=\overline{1,n_j}$ by node S_j; $\mathbf{\Pi}_i\in \mathbf{R}^{n_i\times q}$ is external demand influence matrix. Obviously, a matrix

$$\mathbf{\Pi}=\begin{bmatrix} 0 & \mathbf{\Pi}_{12} & \cdots & \mathbf{\Pi}_{1N} \\ \mathbf{\Pi}_{21} & 0 & \cdots & \mathbf{\Pi}_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{\Pi}_{N1} & \mathbf{\Pi}_{N2} & \cdots & 0 \end{bmatrix} \tag{2}$$

completely characterizes the nodes interactions that define the supply network structure and given technological process.

During the system's operation, the structural constraints should hold:

$$\begin{aligned} \mathbf{x}_i(k)\in X_i &= \left\{\mathbf{x}_i\in \mathbf{R}^{n_i}:\ 0\le \mathbf{x}_i\le \mathbf{x}_i^{\max}\right\}, \\ \mathbf{u}_i(k)\in U_i &= \left\{\mathbf{u}_i\in \mathbf{R}^{m_i}:\ 0\le \mathbf{u}_i\le \mathbf{u}_i^{\max}\right\}, \end{aligned} \tag{3}$$

where the vectors $\mathbf{x}_i^{\max}$ and $\mathbf{u}_i^{\max}$ setting maximal storage capacities of the network nodes and maximal transportation volumes are considered given.

We assume that the vectors of external disturbances satisfy the constraints:

$$\mathbf{d}(k)\in D=\left\{\mathbf{d}\in \mathbf{R}^q:\ \mathbf{d}^{\min}\le \mathbf{d}\le \mathbf{d}^{\max}\right\},$$

where the vectors $\mathbf{d}^{\min}$ and $\mathbf{d}^{\max}$ give the boundary values of demand and are supposed to be known.

Sets of admissible values of the states X_i, controls U_i and demand D are bounded polyhedrons given by intersection of finite number of closed semispaces, i.e. they are compact convex sets, with the coordinate origin being outside their interior: $0 \notin \operatorname{int}(X_i)$, $0 \notin \operatorname{int}(U_i)$, $0 \notin \operatorname{int}(D)$.

For a system consisting of the nodes, the dynamics of which is described by equations (1) and the relationship defined by the matrix (2), we consider the problem of synthesizing a decentralized robust with respect to an unknown, but bounded demand $\mathbf{d}(k) \in D$ inventory control strategy that for any initial state $\mathbf{x}(0)$, where $\mathbf{x}_i(0) \in X_i$, $i = \overline{1, N}$ provides:

- full and timely satisfaction of both external and internal demand;
- minimization of the local quality criteria;
- the asymptotic robust stability of the whole interconnected system;
- fulfillment the given constraints on the state and control (3).

Synthesis of local controllers

Perform the transformation node model (1) to the standard form without delays with extending the state vector [10]:

$$\xi_i(k) = \left[\mathbf{x}_i^{\mathrm{T}}(k), \mathbf{u}_i^{\mathrm{T}}(k-1), \mathbf{u}_i^{\mathrm{T}}(k-2), \ldots, \mathbf{u}_i^{\mathrm{T}}(k - \Lambda_i^{\max})\right]^{\mathbf{T}}.$$

Then the equations of the extended model of the node take the form:

$$\begin{aligned} \xi_i(k+1) &= \mathbf{A}_i \xi_i(k) + \mathbf{B}_i \mathbf{u}_i(k) + \mathbf{G}_i \mathbf{w}_i(k), \\ \mathbf{x}_i(k) &= \mathbf{C}_i \xi_i(k), \end{aligned} \tag{4}$$

where the matrices $\mathbf{A}_i \in \mathbf{R}^{N_i \times N_i}, \mathbf{B}_i \in \mathbf{R}^{N_i \times m_i}, \mathbf{G}_i \in \mathbf{R}^{N_i \times q_i}$, $\mathbf{C}_i \in \mathbf{R}^{n_i \times N_i}$, $N_i = n_i + m_i \Lambda_i^{\max}$ have the respective block structure [9].

Execute approximation of the external actions set for each local node by an ellipsoid of minimum volume. The boundary values of the external actions of network nodes may be found by the following algorithm:

1. $\mathbf{d}_i^{\min} = \mathbf{\Pi}_i \mathbf{d}^{\min}, \quad \mathbf{d}_i^{\max} = \mathbf{\Pi}_i \mathbf{d}^{\max}, \quad i = \overline{1, N}.$

2. $\forall i = \overline{1, q}: \quad \mathbf{\Pi}_i^{\min} = \sum\limits_{j=1,\, j \neq i}^{q} \mathbf{\Pi}_{ij} \mathbf{d}_j^{\min}, \quad \mathbf{\Pi}_i^{\max} = \sum\limits_{j=1,\, j \neq i}^{q} \mathbf{\Pi}_{ij} \mathbf{d}_j^{\max},$

$$\mathbf{w}_i^{\min} = \mathbf{d}_i^{\min} + \mathbf{\Pi}_i^{\min}, \quad \mathbf{w}_i^{\max} = \mathbf{d}_i^{\max} + \mathbf{\Pi}_i^{\max}.$$

3. $\forall i = \overline{q+1, N}: \quad \mathbf{\Pi}_i^{\min} = \sum\limits_{j=1}^{i-1} \mathbf{\Pi}_{ij} \left(\mathbf{\Pi}_j^{\min} + \mathbf{d}_j^{\min}\right), \quad \mathbf{\Pi}_i^{\max} = \sum\limits_{j=1}^{i-1} \mathbf{\Pi}_{ij} \left(\mathbf{\Pi}_j^{\max} + \mathbf{d}_j^{\max}\right),$

$$\mathbf{w}_i^{\min} = \mathbf{d}_i^{\min} + \mathbf{\Pi}_i^{\min}, \quad \mathbf{w}_i^{\max} = \mathbf{d}_i^{\max} + \mathbf{\Pi}_i^{\max}.$$

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Then the external actions set for node S_i can be approximated by ellipsoid:

$$E(\mathbf{w}_i^*, \mathbf{P}_i^w) = \left\{ \mathbf{w}_i \in \mathbf{R}^{n_i} : \left(\mathbf{w}_i(k) - \mathbf{w}_i^*\right)^{\mathrm{T}} \left(\mathbf{P}_i^w\right)^{-1} \left(\mathbf{w}_i(k) - \mathbf{w}_i^*\right) \le 1 \right\}, \quad (5)$$

whose matrix $\mathbf{P}_i^w$ and vector $\mathbf{w}_i^*$ defining the center coordinates are determined by solving a semidefinite programming problem:

$$-\log\det \mathbf{W} \to \min \quad (6)$$

subject to constraints on the matrix $\mathbf{W} = \mathbf{W}^{\mathrm{T}} \in \mathbf{R}^{n_i \times n_i}$ and vector $\mathbf{z} \in \mathbf{R}^{n_i}$ variables:

$$\mathbf{W} \succ 0, \quad \begin{bmatrix} 1 & \left(\mathbf{W}\mathbf{w}_j - \mathbf{z}\right)^{\mathrm{T}} \\ \mathbf{W}\mathbf{w}_j - \mathbf{z} & \mathbf{I}_{n_i \times n_i} \end{bmatrix} \succeq 0, \quad j = \overline{1, 2^{n_i}},$$

where $\mathbf{w}_j$ are the vectors which contain all possible combinations of values of the vectors $\mathbf{w}_i^{\min}$ and $\mathbf{w}_i^{\max}$.

The solution of (6) $\hat{\mathrm{W}}$, $\hat{\mathrm{z}}$ defines the parameters of the ellipsoid (5):

$$\mathrm{P}_i^w = \hat{\mathrm{W}}^{-2}, \quad \mathrm{w}_i^* = \hat{\mathrm{W}}^{-1}\hat{\mathrm{z}}.$$

The local control law is designed in the form of linear non-stationary feedback with respect to the error signal between the available and safety stock levels

$$\mathbf{u}_i(k) = \mathbf{K}_i(k)\left(\xi_i(k) - \xi_i^*\right), \quad (7)$$

where $\mathbf{K}_i(k) \in \mathbf{R}^{m_i \times N_i}$ is the non-stationary feedback gain matrix at the instant k.

The values of the vector ξ_i^* that consists of $\Lambda_i^{\max} + 1$ vectors $\mathbf{x}_i^*$ and gives the amount of safety stock levels are calculated based on the upper boundary values $\mathbf{w}_i^{\max}$ of external actions for the node S_i considering the time-delay value $\Lambda_i^{\max}$:

$$\xi_i^* = \Big[\underbrace{\mathbf{x}_i^{*\mathrm{T}}, \ldots, \mathbf{x}_i^{*\mathrm{T}}}_{\Lambda_i^{\max}+1}\Big]^{\mathrm{T}}, \quad \mathbf{x}_i^* = \Lambda_i^{\max}\mathbf{w}_i^{\max}.$$

Then the extended model of the closed-loop subsystem takes the following form

$$\begin{aligned}&\xi_i(k+1)=\mathbf{A}_{f_i}(k)\left(\xi_i(k)-\xi_i^*\right)+\mathbf{A}_i\xi_i^*+\mathbf{G}_i\left(\mathbf{w}_i(k)-\mathbf{w}_i^*\right)+\mathbf{G}_i\mathbf{w}_i^*,\\&\mathbf{x}_i(k)=\mathbf{C}_i\xi_i(k),\quad \mathbf{A}_{f_i}(k)=\mathbf{A}_i+\mathbf{B}_i\mathbf{K}_i(k).\end{aligned} \tag{8}$$

Local control synthesis problem reduces to the computation of the feedback gain matrices $\mathbf{K}_i(k)$ such that closed subsystems (8) is asymptotically robustly stable. The stability conditions of the whole controlled supply network with decentralized controllers will be discussed below.

Synthesized controller should ensure the minimizing of the following local subsystem criteria in case of an infinite time horizon:

$$J_i^\infty(k)=\sum_{k=0}^{\infty}\left(\left(\xi_i(k)-\xi_i^*\right)^{\mathrm{T}}\mathbf{R}_i^\xi\left(\xi_i(k)-\xi_i^*\right)+\mathbf{u}_i^{\mathrm{T}}(k)\mathbf{R}_i^u\mathbf{u}_i(k)+\Delta\mathbf{u}_i^{\mathrm{T}}(k)\mathbf{R}_i^\Delta\Delta\mathbf{u}_i(k)\right), \tag{9}$$

where $\mathbf{R}_i^\xi\in\mathbf{R}^{N_i\times N_i}$, $\mathbf{R}_i^u\in\mathbf{R}^{m_i\times m_i}$, $\mathbf{R}_i^\Delta\in\mathbf{R}^{m_i\times m_i}$ are diagonal positive definite weighting matrices; $\Delta\mathbf{u}_i(k)=\mathbf{u}_i(k)-\mathbf{u}_i(k-1)$.

The first term in (9) determines the amount of penalties for deviation of available resources levels from safety stock levels, the second one– take into account the cost of resources transportation and storage, and the third – is introduced for smoothing of control actions jumps, since the change in resources production volumes should be carried out smoothly.

Stabilizing control algorithms are generally based on the estimation of the upper boundary value of the system performance criterion using a Lyapunov function. We define the quadratic Lyapunov function, which is built on the subsystem (8) solutions:

$$V_i\left(\xi_i(k)-\xi_i^*\right)=\left(\xi_i(k)-\xi_i^*\right)^{\mathrm{T}}\mathbf{P}_i(k)\left(\xi_i(k)-\xi_i^*\right),\quad \mathbf{P}_i(k)=\mathbf{P}_i^{\mathrm{T}}(k)\succ 0. \tag{10}$$

We require that $\forall k\geq 0$ and any value of the external action belonging approximating ellipsoid $\mathbf{w}_i(k)\in E(\mathbf{w}_i^*,\mathbf{P}_i^w)$ for the first difference of the Lyapunov function computed by k the inequality guaranteeing decrease with time of the function (10) value is occurred:

$$V_i\left(\xi_i(k+1)-\xi_i^*\right)-V_i\left(\xi_i(k)-\xi_i^*\right)\leq -J_i^\infty(k). \tag{11}$$

If inequality (11) holds we can show that $\forall k\geq 0$ the following inequality is valid:

$$V_i\left(\xi_i(k)-\xi_i^*\right)\geq \max_{\mathbf{w}_i(k)\in E(\mathbf{w}_i^*,\,\mathbf{P}_i^w)} J_i^\infty(k). \tag{12}$$

Local control actions $\mathbf{u}_i(k)$ will be determined from the minimization condition of the criterion (9) upper bound. Then in accordance with (12) we find the control actions from the minimization condition of the

Lyapunov function:

$$\mathbf{u}_i(k) = \arg \min_{\mathbf{u}_i(k)\in U_i} V_i\left(\xi_i(k) - \xi_i^*\right). \quad \mathbf{(13)}$$

The problem **(13)** is equivalent to the problem of minimum value computation of a scalar $\gamma_i(k) > 0$ such that $\forall k \geq 0$ the following inequality is valid:

$$\left(\xi_i(k) - \xi_i^*\right)^{\mathrm{T}} \mathbf{P}_i(k)\left(\xi_i(k) - \xi_i^*\right) \leq \gamma_i(k). \tag{14}$$

In accordance with **[11],** introduce the matrix variables

$$\mathbf{Q}_i(k) = \gamma_i(k)\mathbf{P}_i^{-1}(k) \tag{15}$$

and using Schur lemma we present the problems of minimizing the scalar value $\gamma_i(k)$ under the condition **(14)** as a semidefinite programming **(SDP):**

$$\min_{\mathbf{Q}_i(k)} \gamma_i(k)$$

$$\gamma_i(k) > 0, \quad \mathbf{Q}_i(k) \succ 0, \quad \begin{bmatrix} 1 & \left(\xi_i(k) - \xi_i^*\right)^{\mathrm{T}} \\ \left(\xi_i(k) - \xi_i^*\right) & \mathbf{Q}_i(k) \end{bmatrix} \succeq 0. \tag{16}$$

Thus the synthesis problem of local robustly stable control is to calculate at each instant k of the feedback gain matrix $\mathbf{K}_i(k)$ which stabilizes the closed subsystem **(8)** and minimizes the Lyapunov function **(10).** The appropriate results are presented in the following theorem.

T h e o r e m. Consider the subsystem **(4)** with constraints **(3)** which is closed-loop with the control law **(7),** and let the feedback gain matrix $\mathbf{K}_i(k) = \mathbf{Y}_i(k)\mathbf{Q}_i^{-1}(k)$

is obtained by solving the optimization problem

$$\min_{\mathbf{Q}_i(k), \mathbf{Y}_i(k), \alpha_i} \gamma_i(k) \tag{17}$$

subject to constraints on the matrix variables $\mathbf{Q}_i(k)$, $\mathbf{Y}_i(k) \in \mathbf{R}^{m_i \times N_i}$ and scalar parameters $\alpha_i, \gamma_i(k)$**: (16),**

$$\begin{bmatrix}
\mathbf{Q}_i(k) & \mathbf{Y}_i^{\mathrm{T}}(k)\mathbf{R}_i^{\Delta}\mathbf{K}_{k-1} & 0 & 0 & \mathbf{\Sigma}_i^{\mathrm{T}}(k) & 0 & \mathbf{Q}_i(k)\mathbf{R}_i^{\xi\,1/2} & \mathbf{Y}_i^{\mathrm{T}}(k)\mathbf{R}_i^{1/2} \\
\mathbf{K}_{k-1}^{\mathrm{T}}\mathbf{R}_i^{\Delta}\mathbf{Y}_i(k) & \gamma_i(k)\mathbf{K}_{k-1}^{\mathrm{T}}\mathbf{R}_i^{\Delta}\mathbf{K}_{k-1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \left(\mathbf{A}_i - \mathbf{I}\right)^{\mathrm{T}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{G}_i^{\mathrm{T}} & 0 & 0 & 0 \\
\mathbf{\Sigma}_i(k) & 0 & \mathbf{A}_i - \mathbf{I} & \mathbf{G}_i & \mathbf{Q}_i(k) & \gamma_i(k)\mathbf{G}_i & 0 & 0 \\
0 & 0 & 0 & 0 & \gamma_i(k)\mathbf{G}_i^{\mathrm{T}} & \gamma(k)\alpha_i\left(\mathbf{P}_{\mathrm{i}}^{w}\right)^{-1} & 0 & 0 \\
\mathbf{R}_i^{\xi\,1/2}\mathbf{Q}_i(k) & 0 & 0 & 0 & 0 & 0 & \gamma_i(k)\mathbf{I} & 0 \\
\mathbf{R}_i^{1/2}\mathbf{Y}_i(k) & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_i(k)\mathbf{I}
\end{bmatrix} \succeq 0,$$

$$\begin{bmatrix} \mathbf{P}_i^x & \gamma_i(k)\mathbf{C}_i \\ \gamma_i(k)\mathbf{C}_i^{\mathrm{T}} & \mathbf{Q}_i(k) \end{bmatrix} \succeq 0, \qquad \mathbf{Y}_i(k)\left(\xi_{\mathrm{i}}(k)-\xi_i^*\right) \succeq 0\,,$$

$$\begin{bmatrix} \mathbf{Y}_i^{\mathrm{T}}(k)\left(\left(\mathbf{u}_i^{\max}\right)^+\right)^{\mathrm{T}}\left(\xi_i(k)-\xi_i^*\right)^{\mathrm{T}} & \mathbf{Y}_i^{\mathrm{T}}(k)\left(\left(\mathbf{u}_i^{\max}\right)^+\right)^{\mathrm{T}}\left(\xi_i(k)-\xi_i^*\right)^{\mathrm{T}} \\ \left(\xi_i(k)-\xi_i^*\right)\left(\mathbf{u}_i^{\max}\right)^+\mathbf{Y}_i(k) & \mathbf{Q}_i(k) \end{bmatrix} \succeq 0\,,$$

where $\mathbf{\Sigma}_i(k)=\mathbf{A}_i\mathbf{Q}_i(k)+\mathbf{B}_i\mathbf{Y}_i(k)$, $\mathbf{K}_{k-1}=\mathbf{K}_i(k-1)$, $\mathbf{R}_i=\mathbf{R}_i^u+\mathbf{R}_i^{\Delta}$, «$^+$» is Moore-Penrose pseudoinverse, $\mathbf{P}_i^X$ is the matrix of the ellipsoid that approximates the set X_i of admissible state values and is calculated by solving the problem, which is similar to the problem (6).

If the problem (17) which can be viewed as a set of one-dimensional convex optimization problem respect to the parameter α_i and SDP has a solution, then the subsystem (4), which is closed using control law (7), for any initial state $\mathbf{x}_i(0)\in X_i$ under the action of external disturbances $\mathbf{w}_i(k)\in E(\mathbf{w}_i^*, \mathbf{P}_i^w)$ is asymptotically robustly stable under constraints (3).

The proof is analogous to the proof of the Theorem 2 in [12].

Stability analysis of the decentralized supply network control system

If under the decentralized control synthesis for each subsystem S_i the optimization problem (17) is solved, it may be argued that all closed local subsystems are asymptotically robustly stable. In order to analyze the stability of the whole controlled supply network S with decentralized controllers, represent the equations of the extended node model (4), taking into account the relationships (2), by analogy with [5] as follows:

$$\xi_i(k+1)=\mathbf{A}_i\xi_i(k)+\mathbf{B}_i\mathbf{u}_i(k)+\sum_{j=1,\,j\neq i}^{N}\mathbf{B}_{ij}\mathbf{u}_j(k)+\mathbf{F}_i\mathbf{d}(k), \tag{18}$$

where $\mathbf{B}_{ij}{}^{\mathrm{T}}=\left[\mathbf{E}_i\mathbf{\Pi}_{ij} \quad \mathbf{0}_{m_i\times m_j} \quad \cdots \quad \mathbf{0}_{m_i\times m_j}\right]$, $\mathbf{F}_i{}^{\mathrm{T}}=\left[\mathbf{E}_i\mathbf{\Pi}_i \quad \mathbf{0}_{m_i\times q} \quad \cdots \quad \mathbf{0}_{m_i\times q}\right]$.

The dynamic equation (18) under the control (7) takes the form:

$$\xi_i(k+1)=\mathbf{A}_{f\,i}(k)\left(\xi_i(k)-\xi_i^*\right)+\mathbf{A}_i\xi_i^*+\sum_{j=1,\,j\neq i}^{N}\mathbf{F}_{ij}(k)\left(\xi_j(k)-\xi_j^*\right)+\mathbf{F}_i\mathbf{d}(k),$$

where $\mathbf{F}_{ij}{}^{\mathrm{T}}(k)=\left[\mathbf{E}_i\mathbf{\Pi}_{ij}\mathbf{K}_j(k) \quad \mathbf{0}_{m_i\times N_j} \quad \cdots \quad \mathbf{0}_{m_i\times N_j}\right]$

To analyze the stability of the controlled supply network S with decentralized controllers the method of comparison and the mathemati-

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cal tool of vector Lyapunov functions is used [13].

Consider vector Lyapunov function:

$$V\left(\xi(k)-\xi^*\right)=\left[v_1\left(\xi_1(k)-\xi_1^*\right),\ldots,v_N\left(\xi_N(k)-\xi_N^*\right)\right]^{\mathrm{T}}, \qquad (19)$$

where $\xi(k)=\left[\xi_1^{\mathrm{T}}(k),\ldots,\xi_N^{\mathrm{T}}(k)\right]^{\mathrm{T}}$ and $\xi^*=\left[\left(\xi_1^*\right)^{\mathrm{T}},\ldots,\left(\xi_N^*\right)^{\mathrm{T}}\right]^{\mathrm{T}}$ are a composite vectors of appropriate dimension.The components of the function **(19)** is a Lyapunov functions of local subsystems in the Siljak form **[14]**:

$$v_i\left(\xi_i(k)-\xi_i^*\right)=\left(\left(\xi_i(k)-\xi_i^*\right)^{\mathrm{T}}\mathbf{P}_i(k)\left(\xi_i(k)-\xi_i^*\right)\right)^{1/2}, \quad i=\overline{1,N},$$

where the matrices $\mathbf{P}_i(k)$ in accordance with **(15)** are equal $\mathbf{P}_i(k)=\gamma_i(k)\mathbf{Q}_i^{-1}(k)$.

On the basis of function **(19)**, the general Lyapunov function for system S may be formed as follows:

$$V_0\left(\xi(k)-\xi^*\right)=\mathbf{P}_0V\left(\xi(k)-\xi^*\right), \qquad (20)$$

where $\mathbf{P}_0=\left[p_{01},\ldots,p_{0N}\right]$, $p_{0i}>0$, $i=\overline{1,N}$.

Thus the linear comparison system for system defined by difference equations:

$$\upsilon(k+1)=\mathbf{\Lambda}(k)\upsilon(k), \quad \eta(k)=\mathbf{P}_0\upsilon(k), \qquad (21)$$

where $\upsilon=\left[\upsilon_1^{\mathrm{T}},\ldots,\upsilon_N^{\mathrm{T}}\right]^{\mathrm{T}}$ is the state vector of comparison system; η is scalar function, which is the output of comparison system; $\mathbf{\Lambda}(k)\in\mathbf{R}^{N\times N}$ is non-stationary matrix with non-negative elements.

In **[15]** was formulated the theorem according to which for the vector **(19)** and general **(20)** Lyapunov functions following inequalities hold

$$V\left(\xi(k)-\xi^*\right)\le\upsilon(k), \quad V_0\left(\xi(k)-\xi^*\right)\le\eta(k),$$

if the elements of the matrix $\mathbf{\Lambda}(k)$ is determined by characteristic equation of the quadratic forms beam:

$$\begin{aligned}&\det\left(\mathbf{A}_{f\,i}^{\mathrm{T}}(k)\mathbf{P}_i(k)\mathbf{A}_{f\,i}(k)-\mu_{ii}\mathbf{P}_i(k)\right)=0, \quad i=\overline{1,N},\\&\det\left(\mathbf{F}_{ij}^{\mathrm{T}}(k)\mathbf{P}_i(k)\mathbf{F}_{ij}(k)-\mu_{ij}\mathbf{P}_j(k)\right)=0, \quad i,j=\overline{1,N}, \quad j\ne i.\end{aligned} \qquad (22)$$

Herewith $\lambda_{ij}(k)=\left[\mu_{ij}^{\max}\right]^{1/2}$, where $\mu_{ij}^{\max}$ is the maximum value of the appropriate characteristic equation **(22)** root. Thus the comparison system **(21)** majorizes componently vector Lyapunov function **(19)** and gives an upper estimate of the processes behavior of the composite system S. As a result stability analysis of the controlled supply network S with decentralized controllers reduces to the analysis of the comparison system (21).

Under supply network model construction, the nodes are numbered and grouped according to the stages of processing of raw materials and semi-finished products, starting with those that receive external demand. Moreover, any layer of the network combines nodes that are resource suppliers for the nodes belonging to the layers with numbers strictly less than l and at least for one node of the layer $l-1$. As a result, if a directed graph showing the supply network, is a tree, that is, has no cycles, then non-stationary dynamic matrix $\mathbf{\Lambda}(k)$ of the comparison system (21) is lower triangular. Since the diagonal elements values of the matrix $\mathbf{\Lambda}(k)$ are calculated based on the first of equations (22), their values are positive and and no larger than 1: $0<\lambda_{ii}(k)<1$. As a result, non-stationary matrix $\mathbf{\Lambda}(k)\,\forall k$ is nilpotent and therefore, the comparison system (21) is stable. Consequently, the whole controlled supply network S consisting of interconnected subsystems $S_i,\, i=\overline{1,N}$, which are closed by local feedback decentralized controllers (7) is Lyapunov stable.

Conclusions

In this paper the approach to robust decentralized inventory control problem solving in supply networks is proposed. The specific features of this problem are uncertainty, but boundedness of external demand and availability of asymmetric structural constraints on the states and control actions values.

To suppress influence of disturbances that simulate the change of external demand together with ensuring stability of the closed-loop local subsystems, the invariant ellipsoid technique was applied that allowed stating the problem in terms of LMI and reduce the control synthesis to SDP and one-dimensional convex optimization problems. The most important property of obtained solution is the Lyapunov stability of the whole controlled supply network with decentralized controllers which is guaranteed by use of the comparison method and the technique of constructing vector Lyapunov functions.

The resulting control depends on the chosen desirable value of the safety stock levels. One can choose optimal values of local safety levels

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within the proposed technique since the solution of the robust decentralized control synthesis problem involved actually gives the algorithmic dependence between the local safety stock level and the optimal value of the local performance criterion.

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