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F.E. Geche

IMPLEMENTATION OF BOOLEAN FUNCTIONS BY ONE GENERALIZED NEURAL ELEMENT

***Summary**. Present article considers the neural elements (NE) with generalized threshold activation function and represents criterion of Boolean functions implementation on such neural elements.*

***Keywords:** characteristic vector, neural element, activation function, vector structure, recognition, spectrum, character of group.*

Introduction

Developmentofsuitablemethodsforprocessingandsolvingtheproblemof digital signals and images recognition is actual and practically important task. As we know [1-5], selectionofbasisforrepresentationof discrete signals and images is an important stage during their processing and formation of indicators for recognition problem.

Surfaces which are defined by neural elements with conventional threshold activation functions andareusedforseparationofobjectsclasses, arebuiltbymeansofhyperplanes. Neural elements with generalized threshold activation function define nonlinear surfaces andbymeansofthesesurfacesthe complex surfaces are generated, whichcanbesuccessfullyusedforsolving problems of classification and recognition of objects, defined by sets of Boolean vectors.

Thus, developmentofeffectivemethodsverifyingtheimplementationoflogicalgebra functionsby one neural element with generalized threshold activation function is actual and important while solving problems connected with objects recognition and data compression.

Neural elements with generalized threshold activation function

Let $H_2 = \{-1,1\}$ – isa cyclic group of 2-ndorder, $G_n = H_2 \otimes \ldots \otimes H_2$ – isa direct product of n cyclic group H_2 and $X(G_n)$ – is a group of characters [6-8] of group G_n over the field of real numbers R. Letsdefinefunctionon the set $R \setminus \{0\}$:

$$Rsign\, x = \begin{cases} 1, & if\ x > 0, \\ -1, & if\ x < 0. \end{cases} \tag{1}$$

Let $i \in \{0,1,2,\ldots,2^n-1\}$ and $(i_1,\ldots,i_n)$ – is its binary code, that is $i = i_1 2^{n-1} + i_2 2^{n-2} + \ldots + i_n$, $i_j \in \{0,1\}$. The value of character χ_i on element $g = ((-1)^{\alpha_1},\ldots,(-1)^{\alpha_n}) \in G_n$ ($\alpha_j \in \{0,1\}, j = 1,2,\ldots,n$) is defined as follows:

$$\chi_i(g) = (-1)^{\alpha_1 i_1 + \alpha_2 i_2 + \ldots + \alpha_n i_n}.$$

Let us consider 2^n-dimensional vector space $V_R = \{\varphi \mid \varphi : G_n \to R\}$ over the field R. Elements χ_i $(i = 0,1,2,\ldots,2^n-1)$ of group $X(G_n)$ form orthogonal basis of space V_R [6]. Booleanfunctioninthealphabet $\{-1,1\}$ sets definiteindication $f : G_n \to H_2$, that is $f \in V_R$. Consequently, arbitrary Boolean function $f \in V_R$ definitely could be written as follows:

$$f(g) = s_0\chi_0(g) + s_1\chi_1(g) + \ldots + s_{2^n-1}\chi_{2^n-1}(g).$$

Vector $s_f = (s_0, s_1, \ldots, s_{2^n-1})$ isnamedthe spectrum ofBooleanfunctionin system of characters $X(G_n)$ (in system of basis functions of Walsh-Hadamard [9]).

Fromdifferentcharacters $X(G_n)$, exceptthemainone, wewillbuild m-element set $\{\chi_{i_1},\ldots,\chi_{i_m}\}$ and relating the chosen systems of characterswe will consider the following mathematical model of neural element:

$$f(x_1(g),\ldots,x_n(g)) = Rsign\left(\sum_{j=1}^{m}\omega_j\chi_{i_j}(g) + \omega_0\right), \tag{2}$$

wherevector $w = (\omega_1,\ldots,\omega_m;\omega_0)$ isnamedasvectorofneural element structureand $g \in G_n$.

Let $w(g) = \omega_1\chi_{i_1}(g) + \ldots + \omega_m\chi_{i_m}(g) + \omega_0$. If $w = (\omega_1,\ldots,\omega_m;\omega_0)$ isavectorofNEstructurerelatingtosystemofcharacters $\{\chi_{i_1},\ldots\chi_{i_m}\}$ of group G_n over R, which implements Booleanfunction $f : G_n \to H_2$, then from (1) and (2) immediately follows that

$$\forall g \in G_n \quad w(g) \neq 0. \tag{3}$$

Furtherwewillconsideronlysuch neural elements, vectorstructuresofwhich satisfy the condition (3). Setofallsuch $m+1$-dimensional real vectors that satisfy the condition (3), we will mark by means of $W_{m+1} = W_{m+1}(\chi_{i_1},\dots\chi_{i_m})$.

Itisobvious, thatthe neural element relatingtothe system of characters $\{\chi_1, \chi_2, \chi_4, \dots, \chi_{2^n-1}\}$ coincides with the threshold element [10].

Theorem 1. *Boolean function* $f: G_n \to H_2$ *isimplementedbyoneneuralelement relating to system of characters* $\{\chi_{i_1},\dots\chi_{i_m}\} \subset X(G_n)$ *with vector of structure* $\mathrm{w} \in W_{m+1}$ *only in case, when*

$$\forall \mathrm{g} \in G_n \quad f(\mathrm{g})w(\mathrm{g}) = |w(\mathrm{g})|, \tag{4}$$

where $|x|$ *– when module of number* $x \in R$.

The proof follows immediately from (2) and from equality: $(Rsign x)\cdot x = |x|$, where $x \in R \setminus \{0\}$.

Theorem 2. *Boolean function* $f: G_n \to H_2$ *isimplementedbyoneneuralelement relating to system of characters* $\{\chi_{i_1},\dots\chi_{i_m}\} \subset X(G_n)$ *with vector of structure* $\mathrm{w} \in W_{m+1}$ *only in case, when*

$$\sum_{\mathrm{g}\in G_n} f(\mathrm{g})w(\mathrm{g}) = \sum_{\mathrm{g}\in G_n} |w(\mathrm{g})|. \tag{5}$$

Proof. Sufficiency. Indeed, when function f is not implemented by one NE relating $\{\chi_{i_1},\dots,\chi_{i_m}\}$, then such elements exist $\mathrm{g}_1,\dots,\mathrm{g}_k \in G_n$ for which the equality (4) is not true, that is:

$$\forall \mathrm{g}_j \in \{\mathrm{g}_1,\dots,\mathrm{g}_k\} \quad f(\mathrm{g}_j)w(\mathrm{g}_j) = -|w(\mathrm{g}_j)|$$

and

$$\sum_{\mathrm{g}\in G_n} f(\mathrm{g})w(\mathrm{g}) = \sum_{\mathrm{g}\in G_n\setminus\{\chi_{i_1},\dots,\chi_{i_m}\}} |w(\mathrm{g})| - \sum_{j=1}^{k} |w(\mathrm{g}_j)|. \tag{6}$$

Henceon the basisof (5) we have

$$2\sum_{j=1}^{k} |w(\mathrm{g}_j)| = \sum_{\mathrm{g}\in G_n} |w(\mathrm{g})| - \sum_{\mathrm{g}\in G_n} f(\mathrm{g})w(\mathrm{g}) = 0.$$

Consequently, $\forall \mathrm{g}_j \in \{\mathrm{g}_1,\dots,\mathrm{g}_k\}$ $w(\mathrm{g}_j) = 0$, that contradicts the condition $\mathrm{w} \in W_{m+1}$. The necessity follows directly from theorem 1.

Let us write the left side of equality(5) in expanded form:

$$\sum_{g\in G_n} f(g)w(g) = \sum_{g\in G_n} f(g)(\omega_1\chi_{i_1}(g)+\ldots+\omega_m\chi_{i_m}(g)+\omega_0) =$$

$$= \omega_0 \sum_{g\in G_n} f(g) + \sum_{j=1}^{m}\omega_j\left(\sum_{g\in G_n} f(g)\chi_{i_j}(g)\right) = \omega_0 b_0 + \sum_{j=1}^{m}\omega_j b_{i_j},$$

where $\mathrm{b}_f(X) = (b_{i_1},\ldots,b_{i_m};b_0)$, $\left(b_{i_1} = 2^n s_{i_1},\ldots, b_{i_m} = 2^n s_{i_m}, b_0 = 2^n s_0\right)$ – is a characteristicvector of Booleanfunction f relating to the system of characters $X = \{\chi_{i_1},\ldots,\chi_{i_m}\}$.

Usingtheconceptofcharacteristicvectorof Booleanfunction f relatingtothe system of characters $X = \{\chi_{i_1},\ldots,\chi_{i_m}\}$ theorem 2 could be written as follows:

Theorem 3. *Boolean function* $f: G_n \to H_2$ *is implemented by one neural element relating to the system of characters* $X = \{\chi_{i_1},\ldots\chi_{i_m}\} \subset X(G_n)$ *with vector of structure* $\mathrm{w} \in W_{m+1}$ *only in case, when its characteristic vector* $\mathrm{b}_f(X)$ *satisfies the condition*

$$(\mathrm{w}, \mathrm{b}_f(X)) = \sum_{g\in G_n} |w(g)|,$$

where $(\mathrm{w}, \mathrm{b}_f(X))$ – *is a scalar product of vectors* w *and* $\mathrm{b}_f(X)$.

Conclusions

It derives from theorem 3 that Booleanfunction, which is implemented by one neural element relating to system of characters $X = \{\chi_{i_1},\ldots\chi_{i_m}\} \subset X(G_n)$, is definitely determined by its $m+1$-dimensional characteristic vector $\mathrm{b}_f(X) = (b_{i_1},\ldots,b_{i_m};b_0)$, and functions, which are not implemented by one neural element, are definitely determined in spectral region only by its spectrum $\mathrm{s}_f = (s_0, s_1,\ldots,s_{2^n-1})$. Consequently, compression coefficient of Boolean function f, which is implementedbyoneneuralelement in spectral area is set by the ratio $2^n/(m+1)$. This means that characteristic vectors of such Boolean functions could be successfully used for data compression. Maximization of data compression coefficient is reduced to minimization of parameter

$m = m^*$, that is to finding out such minimum system of characters $X^* = \{\chi_{i_1}, \dots \chi_{i_{m^*}}\}$ relating to which the coresponding functions of logic algebra are implemented by one neural element.

If neural element with vector of structure $\mathrm{w} \in W_{m+1}$ relating to system of characters $X = \{\chi_{i_1}, \dots \chi_{i_m}\}$ implements the Boolean function f, then from theorem 3 and equality (6) follows that for Boolean function $h : G_n \to H_2$ $(h \neq f)$ the following inequality is true

$$(\mathrm{w}, \mathrm{b}_f(X)) > (\mathrm{w}, \mathrm{b}_h(X)).$$

The last inequality could be used for construction of iterative procedure of neural elements synthesis with generalized threshold activation function relating to system of characters X.

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