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PACKING OF APPROXIMATED ELLIPSOIDS

The paper considers the problem of packing a given collection of ellipsoids into a container of minimal volume. Our ellipsoids can be continuously rotated and translated. A class of radical free quasi-phi-functions for approximated ellipsoids by polytopes is used for an analytical description of non-overlapping and containment constraints. We formulate the packing problem in the form of a nonlinear programming problem and propose a solution strategy, which allow us to search for local optimal extrema for the packing problem of approximated ellipsoids. These packings may be considered as promising starting points for a packing problem of true ellipsoids. We provide some computational results.

Keywords: *packing, approximated ellipsoids, continuous rotations, non-overlapping, containment, quasi-phi-functions, nonlinear optimization.*

I. INTRODUCTION

In this paper we deal with the optimal ellipsoid packing problem. The problem is NP-hard [1] and has multiple applications in modern biology, mineralogy, medicine, materials science, nanotechnology, as well as in the chemical industry, power engineering etc.

At present, the interest in finding effective solutions for placement problems of ellipsoids is growing rapidly (see, e.g., [2-5]). This is due to a large number of applications and an extreme complexity of methods used to handle many of them.

Paper [6] propose an exact NLP model and a solution strategy for packing spheroids into cuboid of minimal volume using quasi-phi-functions. However the approach allows us to compute feasible solution up to 12 spheroids.

The present paper proposes an approach, which is capable of handling arbitrary ellipsoids approximated by polytopes and thus finding a feasible solution. In addition, an arbitrary convex container, as well as, minimal allowable distances between ellipsoids may be considered.

Our approach reducing the packing problem to a nonlinear programming problem. To this end a class of quasi-phi-functions [7] is used for approximated ellipsoids taking into account their continuous rotations and translations.

To construct a mathematical model of the problem we apply quasi-phi-functions for convex polytopes, derived in [8].

The paper is organized as follows:

In Section 2 we formulate the ellipsoid packing problem.

In Section 3 we present quasi-phi-functions for non-overlapping and containment constraints. In Section 4 we propose a mathematical model as a continuous nonlinear programming problem by means of quasi-phi-functions for approximated ellipsoids and describe a solution strategy. In Section 5 we provide our computational results. Finally we give some conclusions in Section 6.

II. PROBLEM FORMULATION

We consider here a packing problem in the following setting. Let Ω be a cuboid of variable length l, width w and height h. Suppose a set of ellipsoids, E_i, $i \in \{1,2,...,n\} = I_n$, is given to be placed in Ω without overlaps. Each ellipsoid E_i is given by semi-axes a_i and b_i and c_i. With each ellipsoid E_i its local coordinate system whose origin coincides with the center of the ellipsoid and the coordinate axes are aligned with the ellipsoid's axes.

We also use a fixed coordinate system attached to the container Ω. The location and orientation of each ellipsoid E_i is defined by a variable vector of its placement parameters (v_i, θ_i). Here $v_i = (x_i, y_i, z_i)$ is a translation vector, $\theta_i^1, \theta_i^2, \theta_i^3$ are Euler angels in the global coordinate system. The rotated by angles $\theta_i^1, \theta_i^2, \theta_i^3$ and translated by vector v_i ellipsoid E_i is defined as

$$E_i(u) = \{p \in R^3 : p = v_i + M(\theta_i) \cdot p^0, \forall p^0 \in E_i^0\},$$

where E_i^0 denotes the non-translated and non-rotated ellipsoid E_i, $M(\theta)$ is a rotation matrix, where

$$M(\theta) = \begin{pmatrix} \cos\theta^1\cos\theta^3 - \sin\theta^1\cos\theta^2\sin\theta^3 & -\cos\theta^1\sin\theta^3 - \sin\theta^1\cos\theta^2\cos\theta^3 & \sin\theta^1\sin\theta^2 \\ \sin\theta^1\cos\theta^3 + \cos\theta^1\cos\theta^2\sin\theta^3 & -\sin\theta^1\sin\theta^3 + \cos\theta^1\cos\theta^2\cos\theta^3 & -\cos\theta^1\sin\theta^2 \\ \sin\theta^2\sin\theta^3 & \sin\theta^2\cos\theta^3 & \cos\theta^2 \end{pmatrix}.$$

Packing problem of ellipsoids. Pack the set of ellipsoids E_i, $i \in I_n$, into a container Ω of the minimal volume $F = l \cdot w \cdot h$.

III. Quasi-phi-functions for non-overlapping and containment constraints

We approximate each ellipsoid E_i by a convex polytope defined by its vertices p_j^i, $j = 1,....,m$, whose values are fixed in the local coordinate system of the ellipsoid E_i. We denote the approximated ellipsoid by $\hat{E}_i(u_i)$. Now we reformulate our problem *packing problem of ellipsoids* in the following way.

Pack the collection of objects $\hat{E}_i(u_i)$, $i \in I_n$, into a rectangular domain Ω of minimal volume $F = l \cdot w \cdot h$.

Quasi-phi-functions for nonoverlapping constraints. Let $\hat{E}_i(u_i)$ and $\hat{E}_j(u_j)$ be approximated ellipsoids, given by their vertices p_k^i, $k = 1,....,m$, and p_k^j, $k = 1,....,m$.

Let $P(u'_{ij}) = \{(x,y,z) : \psi_P = \alpha_{ij} \cdot x + \beta_{ij} \cdot y + \gamma_{ij} \times z + \mu_{ij} \le 0\}$ be a half-space, where

$$\begin{pmatrix} \alpha_{ij} \\ \beta_{ij} \\ \gamma_{ij} \end{pmatrix} = M(\theta_P^1, \theta_P^2, 0) \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\theta_P^1 & -\sin\theta_P^1 \cos\theta_P^2 & \sin\theta_P^1 \sin\theta_P^2 \\ \sin\theta_P^1 & \cos\theta_P^1 \cos\theta_P^2 & -\cos\theta_P^1 \sin\theta_P^2 \\ 0 & \sin\theta_P^2 & \cos\theta_P^2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \sin\theta_P^1 \sin\theta_P^2 \\ -\cos\theta_P^1 \sin\theta_P^2 \\ \cos\theta_P^2 \end{pmatrix}.$$

Here $u'_{ij} = (\theta_P^1, \theta_P^2, \mu_P)$, θ_P^1 and θ_P^2 are appropriate (precession, nutation rotations) variable Euler angles.

Then, a quasi-phi-function for $\hat{E}_i(u_i)$ and $\hat{E}_j(u_j)$ may be defined as follows

$$\Phi'^{\hat{E}_i \hat{E}_j}(u_i, u_j, u'_{ij}) = \\ = \min\left\{\Phi^{\hat{E}_i P}(u_i, u'_{ij}), \Phi^{\hat{E}_j P^*}(u_j, u'_{ij})\right\}, \quad (1)$$

where $\Phi^{\hat{E}_i P}(u_i, u'_{ij})$ is the normalised phi-function for $\hat{E}_i(u_i)$ and a half-space $P(u'_{ij})$ [21] and $\Phi^{\hat{E}_j P^*}(u_j, u'_{ij})$ is the normalised phi-function for $\hat{E}_j(u_j)$ and

$$P^*(u'_{ij}) = R^3 \setminus \operatorname{int} P(u'_{ij}),$$

where $\Phi^{\hat{E}_i P}(u_i, u'_{ij}) = \min_{1 \le k \le m} \psi_P(p_k^i)$

and $\Phi^{\hat{E}_j P^*}(u_j, u'_{ij}) = \min_{1 \le k \le m} (-\psi_P(p_k^j))$.

Thus a non-overlapping constraint, i.e. $\operatorname{int}\hat{E}_i(u_i) \cap \operatorname{int}\hat{E}_j(u_j) = \varnothing$, can be defined as $\Phi'_{ij}(u_i, u_j, u'_{ij}) \ge 0$, where Φ'_{ij} is a quasi-phi-function for objects $\hat{E}_i(u_i)$ and $\hat{E}_j(u_j)$ given by (1). It is obvious that $\operatorname{int}\hat{E}_i(u_i) \cap \operatorname{int}\hat{E}_j(u_j) = \varnothing$ implies $\operatorname{int} E_i(u_i) \cap \operatorname{int} E_j(u_j) = \varnothing$.

Quasi-phi-functions for containment constraints. Let vertices of cuboid $\Omega = \{(x,y,z) \in R^3 : 0 \le x \le \le l, 0 \le y \le w, 0 \le z \le h\}$ be given: $\{v_i, i = 1,...,8\} = \{(0,w,0), (l,w,0), (l,0,0), (0,0,0), (0,w,h), (l,w,h), (l,0,h), (0,0,h)\}$.

And let $\hat{E}_i(u_i)$ be a convex polytope, given in its local coordinate system by their vertices p_k^i, $k = 1,....,m$, where $p_k^i = (p_{xk}^i, p_{yk}^i, p_{zk}^i)$.

Then a phi-function for objects $\hat{E}_i(u_i)$ and $\Omega^* = R^3 \setminus \operatorname{int}\Omega$ may be defined in the form

$$\Phi^{\hat{E}_i \Omega^*}(u_i) = \min\{\min_{1 \le k \le m} \varphi_{qi}(p_k^i), q = 1,...,6\}. \quad (2)$$

$$\varphi_{1i}(p_k^i) = x_i + p_{xk}^i, \ \varphi_{2i}(p_k^i) = -(x_i + p_{xk}^i) + l,$$

$$\varphi_{3i}(p_k^i) = y_i + p_{yk}^i, \ \varphi_{4i}(p_k^i) = -(y_i + p_{yk}^i) + w,$$

$$\varphi_{5i}(p_k^i) = z_i + p_{zk}^i, \ \varphi_{6i}(p_k^i) = -(z_i + p_{zk}^i) + h.$$

Thus, a containment constraint, i.e.

$$\hat{E}_i(u_i) \subset \Omega \Leftrightarrow \operatorname{int}\hat{E}_i(u_i) \cap \Omega^* = \varnothing,$$

can be defined as $\Phi_i(u_i) \ge 0$, where $\Phi_i(u_i)$ is a phi-function for $\hat{E}_i(u_i)$ and Ω^* given by (2).

IV. Mathematical Model and Solution Strategy

The vector $u \in R^\sigma$ of all our variables can be described as follows: $u = (l, w, h, u_1, u_2, ..., u_n, \tau)$, where (l, w, h) denote the variable dimensions of the container Ω and $u_i = (v_i, \theta_i)$ is the vector of placement parameters for the ellipsoids E_i, $i \in I_n$, where $v_i = (x_i, y_i, z_i)$, $\theta_i = (\theta_i^1, \theta_i^2, \theta_i^3)$. The vector $\tau = (u'_{ij}, i < j = 1,...,n)$ denotes the vector of extra variables (for our quasi-phi-functions), where u'_{ij} is defined in (1).

A mathematical model of the optimal packing problem of approximated ellipsoids may now be stated in the following form:

$$\min_{u \in W \subset R^{\sigma}} F(u), \quad (3)$$

$$W = \left\{ u \in R^{\sigma} : \Phi'_{ij} \geq 0, \Phi_i \geq 0, \quad i, j = \overline{1, n}, \ j > i \right\}, \quad (4)$$

where $F(u) = l \cdot w \cdot h$, $\Phi'^{\hat{E}_i \hat{E}_j}$ is a quasi-phi-function (1) defined for the pair of objects $\hat{E}_i$ and $\hat{E}_j$ to hold *nonoverlapping* constraint (i.e. $\mathrm{int}\,\hat{E}_i \cap \mathrm{int}\,\hat{E}_j \Rightarrow$ $\Rightarrow \mathrm{int}\,E_i \cap \mathrm{int}\,E_j$), Φ_i is phi-function (2) defined for objects $\hat{E}_i$ and Ω^* to hold the *containment* constraint (i.e. $(\hat{E}_i \subset \Omega \Leftrightarrow \mathrm{int}\,\hat{E}_i \cap \mathrm{int}\,\Omega^*) \Rightarrow (\mathrm{int}\,E_i \cap \mathrm{int}\,\Omega^* \Leftrightarrow$ $\Leftrightarrow E_i \subset \Omega)$).

Our constrained optimization problem (3)-(4) is a continuous nonlinear programming problem. We employ the following solution strategy for problem (3) – (4), which involves three major stages, described in [8]:

1. First we generate a number of feasible starting points, using the optimization procedure, which is based on the homothetic object transformations.

2. Then starting from each point obtained at Step 1 we search for a local minimum of the objective function *F*(u) of problem (3)-(4).

3. Lastly, we choose the best local minimum from those found at Step 2. This is our best solution of the problem (3) – (4).

The search for local minima of nonlinear programming problems in our optimization procedures is performed by IPOPT proposed in [9].

Local optimal extreme of problem (3) – (4) may be considered as promising starting points for the optimal packing problem of true ellipsoids.

V. COMPUTATIONAL RESULTS

Here we present a number of Instances to demonstrate the efficiency of our quasi-phi-functions. We have run our experiments on an AMD Athlon 64 X2 5200+ computer. The actual search for a local minimum is performed by IPOPT, which is available at an open access noncommercial software depository (https://projects.coin-or.org/Ipopt) .

We consider a collection of ellipsoids from [6]: $\{E_i, i = 1,..,12\} = \{(a_i, b_i, c_i), i = 1,2,...,12\} = \{$(5, 4, 4), (7, 5, 5), (6, 5, 5), (4, 3, 3), (5.5, 4.5, 4.5), (7.5, 5.5, 5.5), (6.5, 5.5, 5.5), (4.5, 3.5, 3.5), (5.3, 4.3, 4.3), (7.3, 5.3, 5.3), (6.3, 5.3, 5.3), (4.3, 3.3, 3.3)$\}$.

Instance E2. Local optimal placement of objects $\{\hat{E}_i, i = 1,2\}$ is shown in Fig. 1, a. Container has volume $F^* = 2383.620379$ and sizes:

$(l^*, w^*, h^*) = (19.463200, 11.100469, 11.032692)$,

$v_i^* = (x_i^*, y_i^*, z_i^*, \theta_i^{1*}, \theta_i^{2*}, \theta_i^{3*}) = ($3.863703, 6.034755, 4.223965, 1.570796, 2.879792, -3.141593, 13.299281, 5.550235, 5.516346, -2.727019, -1.890560, 5.662936).

$F^* = 2192.513985$ for true ellipsoids.

Instance E3. Local optimal placement of objects $\{\hat{E}_i, i = 1,2,3\}$ is shown in Fig. 1, b. Container has volume $F^* = 3702.936798$ and sizes:

$(l^*, w^*, h^*) = (18.973749, \ 10.639733, \ 18.342664)$.

$v_i^* = (x_i^*, y_i^*, z_i^*, \theta_i^{1*}, \theta_i^{2*}, \theta_i^{3*}) = ($14.346783, 5.326600, 3.863703, -4,500009, -4,500000, -4.500007, 44.829631, 5.319867, 7.016989, 3.403642, -1.614095, 1.582406, 13.665626, 5.319867, 13.035035, -2.702373, -1.973083, -0.694739).

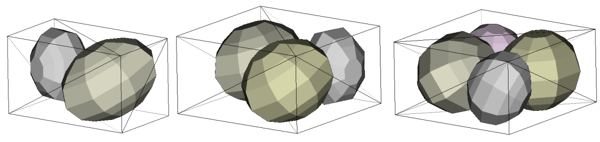
$F^* = 3385.008834$ for true ellipsoids.

Instance E4. Local optimal placement of objects $\{\hat{E}_i, i = 1,...,4\}$ is shown in Fig. 1, c. Container has volume $F^* = 3705.367815$ and sizes:

$(l^*, w^*, h^*) = (18.782844, 10.000000, 19.727405)$.

$v_i^* = (x_i^*, y_i^*, z_i^*, \theta_i^{1*}, \theta_i^{2*}, \theta_i^{3*}) = ($14.422478, 4.904575, 15.863703, -2.058583, -1.832597, -0.000000, 5.251851, 5.000000, 12.672991, 2.143510, -1.361999, 1.439380, 13.782844, 5.000000, 6.000000, -6.283185, -1.570796, -4.712389, 4.434285, 6.033572, 2.970720, 2.924730, 4.454439, -9.429179).

$F^* = 3539.283378$ for true ellipsoids.



a b c

Fig. 1. Local optimal placement of approximated ellipsoids in Instances: a – E2, b – E3, c – E4

Instance E5. Local optimal placement of objects $\{\hat{E}_i, i = 1,...,5\}$ is shown in Fig. 2, a. Container has volume $F^* = 5032.522866$ and sizes:

$(l^*, w^*, h^*) = (19.446479, 19.318523, 13.395869)$.

$v_i^* = (x_i^*, y_i^*, z_i^*, \theta_i^{1*}, \theta_i^{2*}, \theta_i^{3*}) = ($4.674509, 5.542998, 9.532166, 2.279249, -1.832597, 0.000000, 13.884329, 4.829631, 6.697934, -1.289817, 1.203664, 1.467569, 14.013146, 14.488892, 7.490932, -4.429676, 1.188442, -1.678760, 3.506477, 3.703750, 2.963320, -10.287122, -1.414260, -6.283185, 4.678517, 13.315726, 4.458777, 4.221189 , -2.283399, 0.000000).

$F^* = 4347.434370$ for true ellipsoids.

Instance E6. Local optimal placement of objects $\{\hat{E}_i, i = 1,...,6\}$ is shown in Fig. 2, b. Container has volume $F^* = 6805.612826$ and sizes:

$(l^*, w^*, h^*) = (19.943897, 18.491211, 18.454057)$,

$v_i^* = (x_i^*, y_i^*, z_i^*, \theta_i^{1*}, \theta_i^{2*}, \theta_i^{3*}) = ($15.169411, 13.921900, 3.890034, -3.854741, 1.325810, 3.1421361 3.613356, 5.003852, 5.794338, -1.570796, 0.654975, 4.7123891 4.748107, 12.598821, 13.189517, 4.137586, -1.819851, 2.446855, 3.285689, 3.658165, 4.750319, -2.665448, 1.482747, 1.464809 8.237502, 4.686016, 14.013147, -0.478075, -0.884968, 3.141593 5.496256, 13.063430, 7.500000, 2.256626, 1.570796, -4.712389).

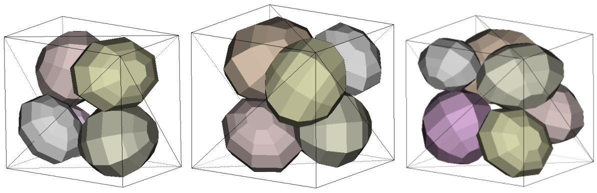
$F^* = 6312.236870$ for true ellipsoids.

Instance E7. Local optimal placement of objects $\{\hat{E}_i, i = 1,...,7\}$ is shown in Fig. 2, c. Container has volume $F^* = 8605.363136$ and sizes:

$(l^*, w^*, h^*) = (22.044030, 20.357230, 19.176066)$,

$v_i^* = (x_i^*, y_i^*, z_i^*, \theta_i^{1*}, \theta_i^{2*}, \theta_i^{3*}) =$ (5.216144, 16.268284, 14.886386, -0.143916, -2.531766, -0.1919161 6.684065, 15.422879, 12.268708, -2.304072, 1.900226, -1.854364, 16.665459, 5.258722, 13.809339, -1.205656, -1.123103, 0.712058 7.597564, 3.671683, 3.866238, -5.109026, -2.085589, 7.01827216.805163, 8.066816, 4.498794, -0.317327, -1.593951, 3.141593 7.226128, 14.253707, 5.312593, 0.452657, -1.308997, -3.141593 5.690613, 6.098289, 13.364751, -2.395971, 1.777532, 2.439335).

$F^* = 7687.512942$ for true ellipsoids.



a b c

Fig. 2. Local optimal placement of approximated ellipsoids in Instances: a – E5, b – E6, c – E7

Instance E8. Local optimal placement of objects $\{\hat{E}_i, i = 1,...,8\}$ is shown in Fig. 3, a. Container has volume $F^* = 9027.867674$ and sizes:

$(l^*, w^*, h^*) = (21.861481, 20.941682, 19.719416)$,

$v_i^* = (x_i^*, y_i^*, z_i^*, \theta_i^{1*}, \theta_i^{2*}, \theta_i^{3*}) =$ (7.080743, 4.283912, 15.732000, 0.490973, 2.183034, -0.0015491 4.879069, 16.096635, 5.369931, -3.739877, -2.838000, -6.903435, 16.967216, 5.893383, 14.338434, 0.801002, -0.449901, 0.8221671 3.874706, 2.898374, 6.883564, 17.547700, -1.355176, -14.1960681 6.634637, 16.183791, 15.035389, 0.566631, 0.618474, -3.619831 6.100744, 6.590548, 6.132972, 0.380940, -0.926308, -2.465021 5.703897, 14.990015, 13.789236, 2.847095, -1.863878, 0.7811341 8.108970, 7.522061, 3.758054, -2.451808, -1.257869 0.661045).

$F^* = 7998.224794$ for true ellipsoids.

Instance E9. Local optimal placement of objects $\{\hat{E}_i, i = 1,...,9\}$ is shown in Fig. 3,b. Container has volume $F^* = 9339.275190$ and sizes:

$(l^*, w^*, h^*) = (1922.399841, 19.597158, 21.275273)$,

$v_i^* = (x_i^*, y_i^*, z_i^*, \theta_i^{1*}, \theta_i^{2*}, \theta_i^{3*}) =$ (12.589207, 5.309804, 4.293675, -0.697499, -0.985668, 2.466853 5.835740, 14.597158, 15.002638, -1.570797, 2.257746, -1.570797, 5.193014, 5.264420, 9.169827, -1491.427003, -1.700896, -1491.2396761 9.502061, 3.195563, 4.011801, -0.262200, -1.516015, -1.55610216.899841, 15.100222, 16.834363, 3.141593, 2.256625, 3.1415931 5.357609, 5.312592, 14.729379, 5.115329, -0.720876, -5.0222921 6.489994, 14.090681, 6.196727, 1.285550, 1.264121, -1.570796 3.848931, 4.406831, 17.882189, 1.926943, 1.312263, -3.141381 4.969423, 14.306685, 4.298287, -0.891367, -1.336103, 0.025524).

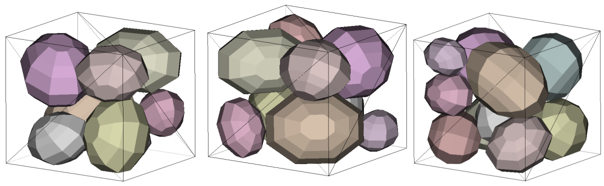
$F^* = 8524.765214$ for true ellipsoids.

Instance E10. Local optimal placement of objects $\{\hat{E}_i, i = 1,...,10\}$ is shown in Fig. 3, c. Container has volume $F^* = 10861.416962$ and sizes:

$(l^*, w^*, h^*) = (22.568674, 21.433929, 22.453222)$,

$v_i^* = (x_i^*, y_i^*, z_i^*, \theta_i^{1*}, \theta_i^{2*}, \theta_i^{3*}) =$ (11.166534, 7.964414, 12.886802, 12.910299, 5.345381, -4.915729 6.479567, 5.341026, 5.501422, 0.715064, 1.268488, 2.5213431 7.572076, 4.934348, 6.042565, 2.256628, -1.570797, -1.570796 4.132997, 18.426593, 19.016688, -3.754969, 2.782015, 2.489901, 17.815779, 4.733867, 17.338235, 12.159119, -0.770725, -11.362496 15.248005, 15.403277, 17.025440, -0.488451, -2.256626, -3.141592 5.427782, 15.720405, 9.671556, -0.740692, 4.359926, 4.406641 4.477619, 11.833159, 18.604429, 0.807084, -0.303720, 2.342486 4.735907, 4.153486, 17.243445, 1.267982, 2.090613, 4.5584301 6.095796, 15.476478, 5.798829, -2.282686, 7.556440, 5.655205).

Average time per one local minimum is 290.23**.** $F^* = 10263.381559$ for true ellipsoids.



a b c

Fig. 3. Local optimal placement of approximated ellipsoids in Instances: a – E8, b – E9, c – E10

Instance E11. Local optimal placement of objects $\{\hat{E}_i, i = 1,...,11\}$ is shown in Fig. 4, a. Container has volume $F^* = 12298.479527$ and sizes:

$(l^*, w^*, h^*) = (28.542303, 19.112259, 22.545007)$,

$v_i^* = (x_i^*, y_i^*, z_i^*, \theta_i^{1*}, \theta_i^{2*}, \theta_i^{3*}) =$ (16.897746, 4.345468, 3.895747, -0.220240, -1.836522, -3.1428741 5.226109, 13.557056, 15.834394, -2.728554, -1.986494, 4.830945, 11.501935, 5.028704, 12.635202, 0.597864, -1.905210, 4.2285542 5.325830, 5.438735, 4.729452, 2.320317, 1.991982, 1.752510 4.602381, 4.467685, 5.577816, 0.925368, -1.642191, 4.6587262 2.576575, 5.312592, 15.106522, -1.302330, 1.350097, 1.51064522.019718, 13.296528, 5.768490, -2.950500, -2.309145, -6.786935 3.499825, 3.928914, 18.109280, -2.563 1.639465, -1.080024, 1.2494112 4.245232, 14.868722, 17.067179, -2.256629, 1.570796, 1.570796 8.352820, 13.746644, 5.246129, 3.183329, 4.024098, 3.150487 5.210337, 13.345436, 16.432640, 2.948000, -1.259914, -1.510897).

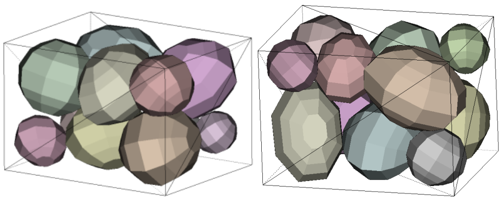
$F^* = 11860.716557$ for true ellipsoids.

Instance E12. Local optimal placement of objects $\{\hat{E}_i, i = 1,...,12\}$ is shown in Fig. 4, b. Container has volume $F^* = 12666.31018$ and sizes:

$(l^*, w^*, h^*) = (28.676637, 20.259273, 21.802086)$,

$v_i^* = (x_i^*, y_i^*, z_i^*, \theta_i^{1*}, \theta_i^{2*}, \theta_i^{3*}) =$ (24.676842, 4.142693, 16.850842, -0.544786, -1.895030, -1.761497 5.320593, 6.686374, 16.547495, -1.023333, 2.536781, -5.8183552 3.580186, 5.971952, 5.082972, -4.832851, 2.208146, -0.0719081 5.174720, 4.153599, 3.006503, -1.552310, 1.325014, 0.007025 4.652385, 15.731921, 5.433617, -4.167647, -1.900265, 4.8896302 1.627998, 14.171563, 16.302086, 2.704467, -1.570796, 0.000000, 6.458752, 5.736486, 5.703897, -2.544192, -1.852385, 0.702256 3.682410, 16.805231, 17.356288, 5.451930, 1.221787, 4.409997 10.341959, 15.536589, 13.038043, 2.981955, -2.144948, -1.6709781 5.772352, 5.243313, 13.725715, -1.805717, -1.479577, 1.6232251 6.686975, 14.458702, 5.479123, 1.348683, -0.686429, 2.4421912 5.435448, 16.504711, 4.112774, -2.913403, -0.980637, -1.699321).

$F^* = 11768.260385$ for true ellipsoids.



a b

Fig. 4. Local optimal placement of approximated ellipsoids in Instances: a – E11, b – E12

VI. Conclusions

We developed here a continuous NLP-model of the packing problem of ellipsoids, using quasi-phi-functions, for polytopes. The use of quasi-phi-functions allows us to handle arbitrary ellipsoids which can be continuously rotated and translated. The model can be realized by the current state-of-the art local or global solvers. The approach allows us to deal with the larger number of arbitrary ellipsoids in comparison with results obtained in [6] for spheroids. In addition arbitrary convex containers and minimal allowable distances between ellipsoids may be considered. Local optimal solutions obtained by our approach may be taken as a promising starting points for a placement of true ellipsoids, using the compaction algorithm, proposed in [7] for ellipses.

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УПАКОВКА АПРОКСИМОВАНИХ ЕЛІПСОЇДІВ

О.М. Хлуд, О.В. Панкратов, Т.Є. Романова

У роботі розглядається задача упаковки заданого набору еліпсоїдів у контейнері мінімального об'єму. Допускаються неперервні трансляції та обертання еліпсоїдів. Для аналітичного описання обмежень неперетину та включення використовуються вільні від радикалів квазі-phi-функції для еліпсоїдів, апроксимованих випуклими багатогранниками. Будується математична модель задачі у вигляді задачі нелінійного програмування (NLP-model) та пропонується стратегія розв'язку, що дозволяє нам отримати локальний екстремум задачі упаковки апроксимованих еліпсоїдів. Результати розв'язання такої задачі можуть бути використані в якості "хорошої" стартової точки для задачі упаковки справжніх еліпсоїдів. Наводяться результати чисельних експериментів.

Ключові слова: *упаковка, апроксимовані еліпсоїди, неперервні обертання, неперетин, включення, квазі-phi-функції, нелінійна оптимізація.*

УПАКОВКА АППРОКСИМИРОВАННЫХ ЭЛЛИПСОИДОВ

О.М. Хлуд, А.В. Панкратов, Т.Е. Романова

В работе рассматривается задача упаковки заданного набора эллипсоидов в контейнере минимального объема. Допускаются непрерывные трансляции и вращения эллипсов. Для аналитического описания ограничений непересечения и включения используются свободные от радикалов квази-phi-функции для эллипсоидов, аппроксимированных выпуклыми многогранниками. Строится математическая модель задачи в виде задачи нелинейного программирования (NLP-model) и предлагается стратегия решения, которая позволяет осуществлять поиск локальных экстремумов для задачи упаковки аппроксимированных эллипсоидов. Такая упаковка может быть использована как "хорошая" стартовая точка для задачи упаковки истинных эллипсоидов. Приводятся результаты численных экспериментов.

Ключевые слова: *упаковка, аппроксимированные эллипсоиды, непрерывные вращения, непересечение, включение, квази-phi-функции, нелинейная оптимизация.*