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GROUP CLASSIFICATION OF THE LINEAR STOCHASTIC DIFFERENTIAL ITO EQUATION

The article deals with the task on the group classification of the linear stochastic differential Ito equation of a given type which changes due to the parameters appearing in this equation. The problem is solved by the symmetry reduction. The result of the study is a table full of group classification of the equations, which lists all the possible equations and allowed their symmetry group.
Keywords: stochastic differential Ito equation, group analysis, commutator, Lie operation algebra.

Introduction. The group analysis of differential equations dates to S. Lie and his pupils' works [1–3]. It was Lie who created and for the first time used the mechanisms of theoretical-group reduction. As a rule, when constructing the mathematical models of real processes they obtain the differential equations which symmetry properties are unknown. That's why of special importance is a technical task of a broader (maximum) group of symmetry admitted by a given differential equation [4, p. 7]. Nowadays the symmetry properties of many known equations of mechanics, gas dynamics, quantum physics (for ex., [5–7]). Many differential equations describing real processes have nontrivial symmetry properties. So, when choosing a differential equation as a mathematical model, symmetry is of a definite value. That makes it possible to effectively use the method of symmetry reduction to construct solutions of such equations. So, owing to the efficiency of using the method of symmetry reduction, it is actual to separate those differential equations out of a given class which have the highest symmetry properties (the task of group classification of differential equations). Solution of this problem is important not only from the mathematical viewpoint, but it is also motivated by a possibility of using the results in different applied problems. The differential equations found in different applied problems often comprise arbitrary parameters and functions. At the same time one can raise a demand for an arbitrary element to have such a form at which an equation admits the broadest invariance group. The full description of the specifications of an arbitrary element makes the sense of the problem of the group classification of differential equations.

Just this very problem is discussed in this article as an addition to the stochastic differential Ito equations. In what follows, the word combination "stochastic differential equation" will be replaced by the abbreviation "SDE".

One of the significant differences of SDE is that their solutions have no derivatives in the classical sense. Thus, the Lie-Ovsyannikov theory cannot be directly applied to the study of the symmetry properties of the Ito SDEs.

The pioneers in developing the theory of the symmetry analysis of SDEs are Yu.L. Daletsky and Ya.I. Belopol'skaya. It was they who introduced the concept of the Ito SDE invariance relative to a one-parameter group of transformations of a phase variable and proved the criterion of the SDE invariance relative to such transformations [8, p. 265]. But the definition of the SDE invariance introduced by the authors of monograph [8] contains the demands which greatly limit the class of groups admitted by SDE. Within such a definition only transformation of a phase variable are admitted and such transformations must be nonrandom functions which do not depend on time. The time variable is not transformed thereat.

Introducing the concept of the SDE invariance, the authors of monograph [8] suppose that these transformations effect only on a phase variable and change only the initial state of the process. Our approach assumes that the group transformations effect both on a phase variable and on a time variable. Here, the Wiener process is not to be maintained and can be transformed into some diffusion process. In this case a germ of the initial diffusion process transforms into a germ of the transformed process in such a way that only their drift and diffusion components must be invariant and the initial state and the Wiener process can be different.

In 2002 S.A. Mel'nick [9] gave the definition of a one-parameter local group of transformations for a stochastic differential equation, but the dependence of the coordinates of the infinitesimal operator of an admissible group on the Wiener process appearing in the equation was not taken into account. In 2004 Italian scientists R. Quinterro and D. Gaeta [10–12] also defined a local one-parameter group for SDEs and proved a corresponding criterion of the equation invariance relative to the admissible group but therewith they did not consider the Wiener process transformation appearing in the equation. That restricted the class of admissible groups for SDEs up to the translation and dilation groups. In [13, 14] they generalized the definition of one-parameter local group of transformations for SDEs given by S.A. Mel'nick, D. Gaeta and R. Quinterro. That made it possible to broaden the class of admissible groups for SDEs. The definition of SDE invariance relative to an admissible group allowed proving the invariance criterion. The invariance criterion of SDEs is a system of linear differential equations in partial derivatives in which the unknown quantities are the coordinates of an admissible group operator, i.e. the system of determining equations. This criterion allows constructing the basic group admitted by the equation. With the help of the proved criterion one can also find the class of equations invariant relative to a given group.

Let's come back to the problem of group classification.

As a basis of the method of classification Lie takes the enumeration of all algebras of the Lie operators. To introduce the concept "Lie algebra", let's determine commutators $[X_1, X_2]$ of any pair of operators of the type

$$X_1 = \xi^{(1)} \frac{\partial}{\partial t} + \eta^{(1)} \frac{\partial}{\partial u}, \quad X_2 = \xi^{(2)} \frac{\partial}{\partial t} + \eta^{(2)} \frac{\partial}{\partial u} \tag{1}$$

with the help of the formula

$$[X_1, X_2] = X_1 X_2 - X_2 X_1 . \tag{2}$$

An operator of the form (1) will result again, that following from the equality:

$$[X_1,X_2]=\left(X_1\left(\xi^{(2)}\right)-X_2\left(\xi^{1}\right)\right)\frac{\partial}{\partial t}+\left(X_1\left(\eta^{(2)}\right)-X_2\left(\eta^{1}\right)\right)\frac{\partial}{\partial u} . \tag{3}$$

Formula (3) can be used instead of (2). [15, p. 10].

Definition 1. [15, p. 10]. Lie operation algebra (1) is called a vector space *L*, which, side by side with any operators $X_1, X_2 \in L$, also includes their commutator $[X_1, X_2]$. This Lie algebra is denoted by *L* and the dimension of algebra is taken as the dimension of vector space *L*.

Problem Statement. Our purpose is the construction of operators allowed by the given equation and construction of Lie operation algebras.

Before stating the results, it should be said that all functions under study are taken as continuous and having continuous derivatives of necessary orders.

Task Solution Design. An ordinary linear stochastic differential Ito equation was considered in complete probabilistic space (Ω, F, P)

$$u(t)=u_0+\int_0^t (bu+\alpha)dh+\int_0^t \left(\sigma u+\beta\right)dW(h), \tag{4}$$

where $t \in [0;T]$, $W(t)$ is a standard Wiener process which is measurable relative to the flow of σ-algebras $\{F_t\}_{t=0}^{\infty}$, $F_t \in F$, $u_0 \in R^1$ and α, b, β, σ are such numbers that task (4) has a unique solution.

Let's formulate and then prove the corresponding theorems.

Theorem 1. Let $b\beta - \sigma\alpha \neq 0, \ \sigma \neq 0$ .Then

1) if $b \neq \sigma^2/2$, then equation (4) admits a two-dimensional algebra of symmetry generated by the operators:

$$X_1 = \partial_t, \quad X_2 = e^{\sigma W(t)+(b-\sigma^2/2)t}\partial_u .$$

2) if $b = \sigma^2/2$, then equation (4) admits a two-dimensional algebra of symmetry generated by the operators:

$$X_1 = \partial_t, \quad X_2 = e^{\sigma W(t)}\partial_u .$$

Theorem 2. Let $b\beta - \sigma\alpha = 0$, $\sigma \neq 0$. Then

1) if $b \neq \sigma^2/2$, $\alpha \neq 0$, $\beta \neq 0$, then Ito equation (4) admits a three-dimensional Lie algebra based on the operators:

$$X_1 = t\partial_t + \frac{\sigma u+\beta}{2\sigma}\left(\ln|\sigma u+\beta| + \left(b-\frac{\sigma^2}{2}\right)t\right)\partial_u, \quad X_2 = \partial_t, \quad X_3 = \left(\sigma u+\beta\right)\partial_u ;$$

2) if $b \neq \sigma^2/2$, $\alpha = 0$, $\beta = 0$, then Ito equation (4) admits a three-dimensional Lie algebra with the basis:

$$X_1 = t\partial_t + \frac{u}{2}\left(\ln|\sigma u| + \left(b-\frac{\sigma^2}{2}\right)t\right)\partial_u, \quad X_2 = \partial_t, \quad X_3 = \sigma u\partial_u ;$$

3) if $b = \sigma^2/2$, $\alpha \neq 0$, $\beta \neq 0$, then Ito equation (4) admits a three-dimensional Lie algebra with the basis:

$$X_1 = t\partial_t + \frac{(\sigma u+\beta)}{2\sigma}\left(\ln|\sigma u+\beta|\right)\partial_u, \quad X_2 = \partial_t, \quad X_3 = (\sigma u+\beta)\partial_u ;$$

4) if $b = \sigma^2/2$, $\alpha = \beta = 0$, then Ito equation (4) admits a three-dimensional Lie algebra with the basis:

$$X_1 = t\partial_t + \left(\frac{u}{2}\ln|\sigma u|\right)\partial_u, \quad X_2 = \partial_t, \quad X_3 = \sigma u \partial_u .$$

Theorem 3. Assume that in equation (4) σ = 0. Then:
1) at $b \neq 0$, $\alpha \neq 0$ and $\beta \neq 0$ it admits a four-parameter group generated by the operators:

$$X_1 = \frac{e^{2bt}}{2b}\partial_t + \frac{e^{2bt}(bu+\alpha)}{2b}\partial_u, \quad X_2 = \partial_t, \quad X_3 = \frac{2e^{bt}}{b}\partial_t + (2u - \beta W(t))e^{bt}\partial_u, \quad X_4 = e^{bt}\partial_u .$$

2) at $b \neq 0$, $\alpha = 0$, $\beta \neq 0$ it admits a four-parameter group generated by the operators:

$$X_1 = \frac{e^{2bt}}{2b}\partial_t + \frac{e^{2bt}}{2}u\partial_u, \quad X_2 = \partial_t, \quad X_3 = \frac{2e^{bt}}{b}\partial_t + (2u - \beta W(t))e^{bt}\partial_u, \quad X_4 = e^{bt}\partial_u .$$

Theorem 4. If in equation (4) σ = 0, $b \neq 0$, $\alpha \neq 0$ and β = 0, it admits the infinite Lie algebra:

$$X_1 = \xi(t)\partial_t + (bu+\alpha)\xi(t)\partial_u, \quad X_2 = e^{bt}\partial_u, \quad X_3 = \partial_t .$$

Theorem 5. Assume that in equation (4) σ = 0, b = 0. Then:
1) if $\alpha \neq 0$, $\beta \neq 0$, then it admits a three-dimensional Lie algebra with the basis:

$$X_1 = 2t\partial_t + (u + \alpha t)\partial_u, \quad X_2 = \partial_t, \quad X_3 = \partial_u .,$$

2) if $\alpha = 0$, $\beta \neq 0$, then it admits a three-dimensional Lie algebra with the basis:

$$X_1 = \partial_t, \quad X_2 = \partial_u, \quad X_3 = 2t\partial_t + u\partial_u .$$

On its own let's consider a special case when the expression under logarithm in theorem 2 vanishes. Let's formulate this case in the form of the following theorem.

Theorem 6. If in equation (4) $\sigma \neq 0$, $b\beta - \sigma\alpha = 0$, and $u_0 = -\beta/\sigma$, then $u(t) \equiv -\beta/\sigma$.

Proving. Assume that in equation(4) $u_0 = -\beta/\sigma$, then:

$$u(t) = -\frac{\beta}{\sigma} + \int_0^t \left(-\frac{b\beta}{\sigma} + \alpha\right)dh + \int_0^t \left(-\frac{\sigma\beta}{\sigma} + \beta\right)dW(h), \tag{5}$$

whence if $b\beta - \sigma\alpha = 0$, then there appears the equality $u(t) = -\beta/\sigma$.

Theorem 6 is proved.

Corollary fact. If in equation (4) $\sigma \neq 0$, $b\beta - \sigma\alpha = 0$, β = 0 and $u_0 = 0$, then $u(t) = 0$.

Proving. If $\beta = 0$, then the approval of the corollary fact results from theorem (6) if $\beta = 0$ is substituted into formula (5).

Comment 1. The expression under logarithm in theorem 2 vanishes only if $u_0 = -\beta/\sigma$. In proving theorem 2 we'll suppose that $u_0 \neq -\beta/\sigma$. Then at $\beta = 0$ (points 2) and 4) of theorem 2) it's naturally to assume that $u_0 \neq 0$.

Comment 2. Formulated theorems (1)–(5) let us divide the proving into two moments which will refer to cases $\sigma = 0$ and $\sigma \neq 0$.

Case $\sigma \neq 0$. An admissible operator of equation (4) will be sought for in form [14]:

$$X = \xi(t)\partial_t + \eta(t, W(t), u)\partial_u .$$

To calculate the operator coordinates coefficients $A(t,u) = bu + \alpha$, $B(t,u) = \sigma u + \beta$ are to be substituted into the system of determining equations [14]:

$$\begin{cases} -\eta_W + \left(\frac{1}{2}\xi_t - \eta_u\right)(\sigma u + \beta) + \eta\sigma = 0, \\ -\eta_t + (\xi_t - \eta_u)(bu + \alpha) + b\eta - \frac{1}{2}\eta_{WW} - \eta_{Wu}(\sigma u + \beta) - \frac{1}{2}\eta_{uu}(\sigma u + \beta)^2 = 0. \end{cases} \tag{6}$$

The general solution of the first equation of the obtained system (6) is the function:

$$\eta(t, W(t), u) = \frac{\xi_t}{2\sigma}(\sigma u + \beta)\ln|\sigma u + \beta| + (\sigma u + \beta)F\left(t, \frac{e^{\sigma W(t)}}{\sigma u + \beta}\right). \tag{7}$$

Let

$$z = e^{\sigma W(t)} \big/ (\sigma u + \beta). \tag{8}$$

then $F(t,z)$ is an arbitrary function continuously differentiated by variables *t* and *z*.

Having substituted the computed η into the second equation of system (6), we'll obtain the equation:

$$-\frac{\xi_{tt}}{2\sigma}(\sigma u+\beta)\ln|\sigma u+\beta|-\frac{\xi_t}{2}(bu+\alpha)\ln|\sigma u+\beta|+\frac{b\xi_t}{2\sigma}(\sigma u+\beta)\ln|\sigma u+\beta|+$$
$$+\frac{1}{2}\left(bu+\alpha-\frac{\sigma(\sigma u+\beta)}{2}\right)\xi_t-(\sigma u+\beta)F_t-\sigma(bu+\alpha)F+$$
$$+b(\sigma u+\beta)F+\left(\sigma(bu+\alpha)\frac{e^{\sigma W(t)}}{(\sigma u+\beta)}-\frac{\sigma^2 e^{\sigma W(t)}}{2}\right)F_z=0. \quad (9)$$

Let

$$\delta=\alpha\sigma-\beta b, \quad \gamma=b-\sigma^2/2. \quad (10)$$

Rewrite the last equation (9), with equations (8) and (10) being taken into account:

$$-\frac{\xi_{tt}}{2\sigma}\cdot\frac{e^{\sigma W(t)}}{z}\cdot\ln\left|\frac{e^{\sigma W(t)}}{z}\right|-\frac{\delta}{2\sigma}\cdot\ln\left|\frac{e^{\sigma W(t)}}{z}\right|\cdot\xi_t+\frac{\xi_t}{2\sigma}\left[\gamma\frac{e^{\sigma W(t)}}{z}+\delta\right]-$$
$$-\frac{e^{\sigma W(t)}}{z}F_t-\delta F+\left[\gamma e^{\sigma W(t)}+\delta z\right]F_z=0. \quad (11)$$

Equation (11) is a classifying one as it comprises the arbitrary elements δ and γ. Depending on either these elements are equal to zero or not we'll obtain the result formulated in theorems (1) and (2).

Formation of symmetry of a linear stochastic differential Ito equation in the case $\sigma = 0$. In the case $\sigma=0$ equation (4) will take the form:

$$u(t)=u_0+\int_0^t(bu(h)+\alpha)dh+\beta\int_0^t dW(h). \quad (12)$$

To calculate the coordinates of the admissible operator, substitute coefficients $A(t,u)=bu+\alpha$ and $B(t,u)=\beta$ into the system of the determining equations [14]:

$$\begin{cases}-\eta_W+\left(\frac{1}{2}\xi_t-\eta_u\right)\beta=0,\\ -\eta_t+(\xi_t-\eta_u)(bu+\alpha)+b\eta-\frac{1}{2}\eta_{WW}-\eta_{Wu}\beta-\frac{1}{2}\eta_{uu}\beta^2=0.\end{cases} \quad (13)$$

The general solution of the first equation of the obtained system is the function:

$$\eta(t,W(t),u)=\frac{1}{2}\xi_t u+\Phi(t,u-\beta W(t)), \quad (14)$$

where Φ is an arbitrary continuously differentiable function which is to be determined.

Let

$$r=u-\beta W(t), \quad (15)$$

then $\Phi(t,u-\beta W(t))=\Phi(t,r)$. Substitute the obtained expression for $\eta(t,W(t),u)$ into the second equation of system (13) and as a result we'll obtain:

$$-\frac{1}{2}\xi_{tt}u-\Phi_t+(bu+\alpha)\left(\frac{1}{2}\xi_t-\Phi_r\right)+\frac{bu}{2}\xi_t+b\Phi=0.$$

Taking into account the symbols of (15), the last equation can be rewritten in the form:

$$-\frac{1}{2}\xi_{tt}(r+\beta W(t))-\Phi_t+\left(b(r+\beta W(t))+\alpha\right)\left(\frac{1}{2}\xi_t-\Phi_r\right)+\frac{b(r+\beta W(t))}{2}\xi_t+b\Phi=0. \quad (16)$$

Equation (16) is a a classifying one as it comprises the arbitrary elements *α, β* and *b*. Depending on either these elements are equal to zero or not we'll obtain the result formulated in theorems (3)–(5).

Conclusions. There has been made a group classification of the ordinary linear stochastic differential Ito equation of the general form depending on combinations of the parameters constituent of the given equation.

The main results are given in the Table.

Table

Group classification of the linear stochastic differential Ito equation

Group	Basic operators	Equation
G_2	$X_1=\partial_t,\ X_2=e^{\sigma W(t)+(b-\sigma^2/2)t}\partial_u$.	$du(t)=(bu+\alpha)dt+(\sigma u+\beta)dW(t)$, $b\beta-\sigma\alpha\neq 0,\ \sigma\neq 0$.
G_3	$X_1=\partial_t$, $X_2=t\partial_t+\frac{\sigma u+\beta}{2\sigma}\left(\ln(\sigma u+\beta)+\left(b-\frac{\sigma^2}{2}\right)t\right)\partial_u$ $X_3=(\sigma u+\beta)\partial_u$.	$du=\frac{\alpha}{\beta}(\sigma u+\beta)dt+(\sigma u+\beta)dW(t)$. $b\beta-\sigma\alpha=0,\ \sigma\neq 0$.
	$X_1=\partial_t,\ X_2=t\partial_t+\frac{u}{2}\left(\ln(\sigma u)+\left(b-\frac{\sigma^2}{2}\right)t\right)\partial_u$, $X_3=\sigma u\partial_u$.	$du(t)=budt+\sigma udW(t)$.
	$X_1=\partial_t,\ X_2=t\partial_t+\frac{(\sigma u+\beta)}{2\sigma}\ln(\sigma u+\beta)\partial_u$, $X_3=(\sigma u+\beta)\partial_u$.	$du(t)=\left(\frac{\sigma^2}{2}u+\alpha\right)dt+(\sigma u+\beta)dW(t)$.
	$X_1=\xi(t)\partial_t,\ X_2=(bu+\alpha)\xi(t)\partial_u$, $X_3=e^{bt}\partial_u$.	$du(t)=(bu+\alpha)dt$.
	$X_1=\partial_t,\ X_2=2t\partial_t+(u+\alpha t)\partial_u$, $X_3=\partial_u$.	$du(t)=\alpha dt+\beta dW(t)$.
	$X_1=\partial_t,\ X_2=\partial_u,\ X_3=2t\partial_t+u\partial_u$.	$du(t)=\beta dW(t)$.
G_4	$X_1=\partial_t,\ X_2=\frac{e^{2bt}}{2b}\partial_t+\frac{e^{2bt}(bu+\alpha)}{2b}\partial_u$, $X_3=\frac{2e^{bt}}{b}\partial_t+(2u-\beta W(t))e^{bt}\partial_u,\ X_4=e^{bt}\partial_u$.	$du(t)=(bu+\alpha)dt+\beta dW(t)$.
	$X_1=\partial_t,\ X_2=\frac{e^{2bt}}{2b}\partial_t+\frac{e^{2bt}}{2}u\partial_u$, $X_3=\frac{2e^{bt}}{b}\partial_t+(2u-\beta W(t))e^{bt}\partial_u,\ X_4=e^{bt}\partial_u$.	$du(t)=budt+\beta dW(t)$.

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РЕЗЮМЕ

Рассматривается задача о групповой классификации линейного стохастического дифференциального уравнения Ито заданного вида, которое изменяется за счет параметров, входящих в это уравнение. Поставленная задача решена методом симметрийной редукции. Результатом проведенного исследования является таблица полной групповой классификации рассмотренного уравнения, в которой приведены все возможные уравнения и допускаемые ими группы симметрии.

Ключевые слова: стохастическое дифференциальное уравнение Ито, групповой анализ, коммутатор, алгебра Ли операторов.

РЕЗЮМЕ

Розглядається задача про групову класифікацію лінійного стохастичного диференціального рівняння Іто заданого виду, яке змінюється за рахунок параметрів, що входять в це рівняння. Поставлена задача вирішена методом симетрійної редукції. Результатом проведеного дослідження є таблиця повної групової класифікації розглянутого рівняння, в якій наведено всі можливі рівняння і допустимі ними групи симетрії.

Ключові слова: стохастичне диференціальне рівняння Іто, груповий аналіз, комутатор, алгебра Лі операторів.