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S.S. Antropov

Oles Honchar Dnipropetrovsk National University

FITTING OF BINDER CUMULANTS IN SU(2) - GLUODYNAMICS

The Binder cumulants are calculated and analyzed in SU(2) lattice gluodynamics. The Binder cumulant is lattice observable quantity, which is constructed out of powers of the Polyakov loop. It describes different aspects of the lattice theories and equals to scaling function in case of SU(2) theory. Such computations become possible due to a technology of calculations on the graphics processing unit (GPU). GPU is used as a computing platform allowing a huge amount of statistical data to be treated over a short period of time. The statistics gathered allows the study of Binder cumulants for a great number of various lattices. Main features of cumulants fitting are described and discussed in detail. The cumulant fitting function is proposed which is based on analysis of obtained data. A few data points (10 up to 20) are sufficient for computation of this function. The important feature of the function constructed is ability to estimate quickly the critical value of the inverse coupling constant β_C on a lattice. The procedure of determination of the intersection point of Binder's cumulants, which should cross in one point, is considered. The updated fitting procedure is proposed for determination of such cumulants. The application of the results is discussed.

Keywords: SU(2) gluodynamics, lattice gauge theory, Monte-Carlo calculations, deconfinement phase transition.

В SU(2) глюодинамике на решетке вычисляются и анализируются кумулянты Биндера. Кумулянт Биндера является наблюдаемой на решетке величиной, которая строится из степеней петли Полякова. Он описывает различные аспекты решеточных теорий и в случае SU(2) теории совпадает с масштабной функцией. Такие расчеты стали возможными благодаря технологии вычислений на графических процессорах. Графических процессоры используются в качестве вычислительной платформы, что позволяет получать большое количество статистических данных за короткий промежуток времени. Собранная статистика делает возможным изучение кумулянтов Биндера на большом количестве различных решеток. Описаны и детально обсуждаются основные особенности фитирования кумулянтов. Опираясь на анализ полученных данных, предложена фитирующая функция для кумулянта. Для вычисления этой функции достаточно несколько точек (от 10 до 20). Важная особенность построенной функции заключается в возможности быстро оценить критические значения обратной константы связи β_C на решетке. Рассмотрена процедура нахождения точки пересечения тех кумулянтов Биндера, которые должны пересекаться в одной точке. Обсуждается применение полученных результатов.

Ключевые слова: SU(2) глюодинамика, калибровочная теория на решетке, Монте-Карло вычисления, фазовый переход к деконфайменту.

У SU(2) глюодинаміці на ґратці обчислюються та аналізуються кумулянти Біндера. Кумулянт Біндера є величиною, що спостерігається на ґратці та яка побудована із ступенів петлі Полякова. Він описує різноманітні аспекти ґраткових теорій та у випадку SU(2) теорії співпадає з масштабною функцією. Такі обчислення стали можливими завдяки технології розрахунків на графічних процесорах. Графічні процесори використовуються у якості обчислювальної платформи, що дозволяє отримувати велику кількість статистичних даних за короткі проміжки часу. Зібрана статистика робить можливим дослідження кумулянтів Біндера на великій кількості різноманітних ґраток. Описані та детально обговорюються основні особливості фітування кумулянтів. Спираючись на аналіз отриманих даних, пропонується фітуюча функція для кумулянтів. Для обчислення цієї функції достатньо декілька точок (від 10 до 20). Важлива особливість запропонованої функції полягає у можливості швидко оцінити критичне значення оберненої константи зв'язку β_C на ґратці. Розглянуто процедуру знаходження точки перетину тих кумулянтів Біндера, які повинні перетинатись у одній точці. Обговорюються застосування отриманих результатів.

Ключові слова: SU(2) глюодинаміка, калібрувальна теорія на ґратці, Монте-Карло розрахунки, фазовий перехід до деконфайменту.

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Introduction

The Polyakov loop is quantity of interest in Monte-Carlo (MC) calculations in the lattice gluodynamics. In particular, it was used for calculation of critical indexes in SU(2)-theory [1]. The value of the Polyakov loop is an important order parameter of SU(N)-theories. It reads

$$P = \frac{1}{N_\sigma^3} \sum_{\bar{x}} \frac{1}{2} Tr \prod_{\tau=1}^{N_\tau} U_{\tau,\bar{x}} \tag{1}$$

where N_σ is the number of lattice sites in each spatial direction, N_τ is the number of lattice sites in the time direction, the summation is assumed over all the spatial coordinates of lattice sites $\bar{x}$. There are such quantities, which can be constructed out of powers of the Polyakov loop and describe different aspects of the theory. One of them is the Binder cumulant [2]

$$g_4 = \frac{\left\langle P^4 \right\rangle}{\left\langle P^2 \right\rangle^2} - 3 \tag{2}$$

where brackets $\langle .. \rangle$ mean the averaging over MC configurations. The particular interest of studying (2) has arisen after ref. [2]. Such cumulant identically coincides with the scaling function of SU(2)-theory [2] and has been used for the critical temperature calculation [3, 4].

In present paper we analyze Binder cumulants. Such cumulants are calculated on various lattices to demonstrate features of fitting of the cumulants.

Lattice computations of the Binder cumulant

The Binder cumulants are investigated in SU(2)-gluodynamics on the lattice. Computer modeling is carried out using Monte-Carlo method. In the MC simulations, we use the hypercubic lattice $N_\tau \times N_\sigma^3$ with hypertorus geometry. The spatial part of the lattice is cubic. The main features of the chosen MC procedure are listed below. We chose the heat-bath as working algorithm in MC procedure. We use standard form of the Wilson action of the SU(2)-lattice gauge theory. We use the thermalization procedure to generate initial conditions for MC calculations. We chose common values for MC parameters: 200 thermalizing sweeps, then 1000 working MC iterations [5, 6]. We set up to 8 MC attempts for MC updating of each of lattice variables [5].

A few words should to say about our computing platform. We use the General Purpose computation on Graphics Processing Units (GPGPU) technology allowing studying large lattices on personal computers. The performance analysis indicates that the GPU-based MC simulation program shows better speed-up factors for big lattices in comparison with the CPU-based one. The GPU vs. CPU (single-thread CPU execution) speed-up factor is above 50 for the majority of lattice geometries and for some lattice sizes can overcome the factor 100 [7]. We use the video controller of the personal computer with GPU manufactured by ATI Company. The programming language is ATI CAL. Because of the features of this language and the GPU architecture we use lattices with even time part and the number of the sites in each of the spatial directions is always multiple of four. Calculations are carried out for lattices with the parameter N_τ, which is varied from 2 up to 16, and parameter N_σ has the values changing from 8 to 32. The number of the fitting points of the dependence of the Binder cumulant g_4 on β is varied for each lattice from 26 to 600.

The result of calculations of (2) is the set of the points. We put the accent on the functional dependence which describes the Binder cumulant both in critical region and beyond. To identify this relationship, it is necessary to apply a fitting procedure. For this procedure we use the step functions. Let us explain, the g_4 cumulant has two horizontal asymptoticses, and in the critical area cumulant changes itself step-like. The next function (see Tab. 1) has the smallest parameter χ^2 and the best fits for the dependence $g_4(\beta)$ in the critical and beyond critical areas reads

$$g_4(\beta) = A_1 + \frac{A_2 - A_1}{1+10^{(\beta_0-\beta)\times p}} \tag{3}$$

where A_1, A_2, β_0, p are the fitting parameters. The results of the fitting of the function (3) are given in the Fig. 1 and on the Tab. 2.

Table 1

Tested fitting curves

Function	Parameters
$A_1 + \frac{A_2 - A_1}{1+10^{(\beta_0-\beta)\times p}}$	A_1, A_2, β_0, p
$\frac{A_1 - A_2}{1+\left(\frac{\beta}{\beta_0}\right)^p} + A_2$	A_1, A_2, β_0, p
$\frac{A_1 - A_2}{1+e^{(\beta-\beta_0)/p}} + A_2$	A_1, A_2, β_0, p

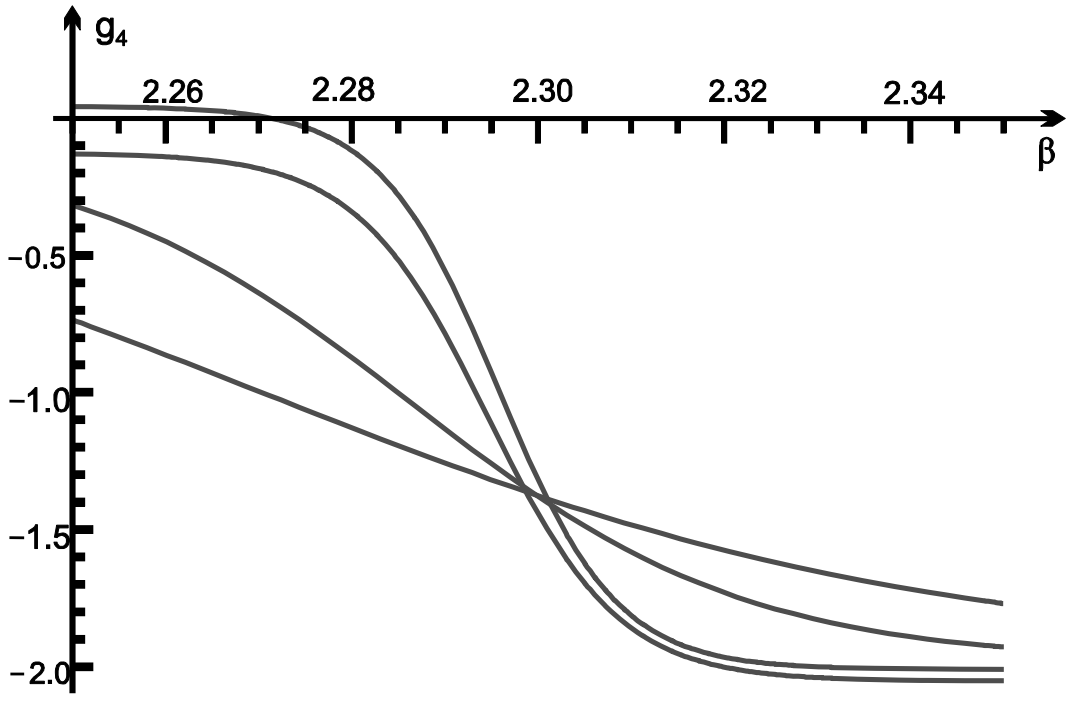

Figure 1. Binder cumulants. The cumulants are calculated on lattices with $N_\tau = 4$ and $N_\sigma = 8, 12, 24, 28$. The higher number of nods in the lattice corresponds with the sharper step.

Table 2

Fitting of Binder cumulants by $A_1 + \dfrac{A_2 - A_1}{1+10^{(\beta_0-\beta)\times p}}$

Lattice	Parameters					Number of points	Fitting range	
	χ^2	A_1	A_2	β_0	p		$\beta_{\min}$	$\beta_{\max}$
$N_\tau = 2, N_\sigma = 8$	0.006	−2	−0.13	1.86	−18	126	1.7	2.95
$N_\tau = 2, N_\sigma = 12$	0.013	−2	−0.08	1.86	−34	126	1.7	2.95
$N_\tau = 2, N_\sigma = 16$	0.013	−2	−0.16	1.87	−43	126	1.7	2.95
$N_\tau = 2, N_\sigma = 20$	0.015	−2	−0.11	1.87	−81	126	1.7	2.95
$N_\tau = 2, N_\sigma = 24$	0.015	−2	−0.28	1.87	−117	126	1.7	2.95
$N_\tau = 2, N_\sigma = 28$	0.008	−2	−0.03	1.87	−77	123	1.7	2.95
$N_\tau = 2, N_\sigma = 32$	0.006	−2	0.14	1.86	−63	124	1.7	2.95
$N_\tau = 4, N_\sigma = 8$	0.009	−1.953	−0.0523	2.2705	−12	126	1.7	2.95
$N_\tau = 4, N_\sigma = 8$	0.012	−1.957	−0.0507	2.2747	−11	26	1.7	2.95
$N_\tau = 4, N_\sigma = 12$	0.025	−1.98	−0.1	2.286	−24	253	1.7	2.95
$N_\tau = 4, N_\sigma = 12$	0.011	−2	−0.04	2.289	−16	26	1.7	2.95
$N_\tau = 4, N_\sigma = 16$	0.029	−2.01	−0.066	2.287	−30.1	236	1.7	2.95
$N_\tau = 4, N_\sigma = 16$	0.013	−1.99	−0.05	2.292	−30.9	26	1.7	2.95
$N_\tau = 4, N_\sigma = 20$	0.055	−2	−0.065	2.291	−48	246	1.7	2.95
$N_\tau = 4, N_\sigma = 24$	0.1	−2.0098	0.044	2.296	−68	126	1.7	2.95
$N_\tau = 4, N_\sigma = 24$	0.006	−2.001	0.061	2.291	−27	26	1.7	2.95
$N_\tau = 4, N_\sigma = 28$	0.089	−2.05	−0.13	2.29	−62	626	1.7	2.95
$N_\tau = 4, N_\sigma = 28$	0.012	−1.99	$-8\cdot10^{-5}$	2.28	−21	26	1.7	2.95
$N_\tau = 4, N_\sigma = 32$	0.12	−1.984	−0.2	2.3	−84	626	1.7	2.95
$N_\tau = 4, N_\sigma = 32$	0.01	−1.988	0.014	2.27	−28	26	1.7	2.95
$N_\tau = 4, N_\sigma = 36$	0.19	−2	−0.27	2.3	−105	600	2.28	2.31
$N_\tau = 6, N_\sigma = 8$	0.014	−1.65	−0.067	2.4	−10.5	127	1.7	2.95
$N_\tau = 6, N_\sigma = 12$	0.025	−1.9	0.05	2.4	−17	127	1.7	2.95
$N_\tau = 6, N_\sigma = 16$	0.032	−2	−0.04	2.4	−17	127	1.7	2.95
$N_\tau = 6, N_\sigma = 20$	0.092	−2	−0.02	2.4	−44	126	1.7	2.95
$N_\tau = 6, N_\sigma = 24$	0.14	−2	−0.04	2.4	−37	127	1.7	2.95
$N_\tau = 6, N_\sigma = 28$	0.2	−2	−0.1	2.4	−41	127	1.7	2.95
$N_\tau = 6, N_\sigma = 32$	0.04	−2	$7\cdot10^{-4}$	2.4	−200	26	1.7	2.95
$N_\tau = 8, N_\sigma = 12$	0.023	−1.8	−0.07	2.48	−11	126	1.7	2.95
$N_\tau = 8, N_\sigma = 16$	0.05	−1.9	0.005	2.49	−13	126	1.7	2.95
$N_\tau = 8, N_\sigma = 20$	0.06	−2	$5\cdot10^{-4}$	2.48	−13	126	1.7	2.95
$N_\tau = 8, N_\sigma = 24$	0.14	−2	−0.0014	2.5	−34	127	1.7	2.95
$N_\tau = 8, N_\sigma = 28$	0.022	−1.9	−0.06	2.49	−26	26	1.7	2.95
$N_\tau = 8, N_\sigma = 32$	0.0115	−2	−0.02	2.48	−15	26	1.7	2.95
$N_\tau = 16, N_\sigma = 20$	0.094	−1.17	−0.017	2.68	−7	126	1.7	2.95
$N_\tau = 16, N_\sigma = 24$	0.054	−1.7	0.04	2.75	−6	26	1.7	2.95
$N_\tau = 16, N_\sigma = 28$	0.021	−1.6	−0.017	2.67	−17	26	1.7	2.95
$N_\tau = 16, N_\sigma = 32$	0.021	−1.7	0.03	2.69	−23	126	1.7	2.95

If one knows the dependencies (3) for various lattices, it is easy to find the critical value of the inverse critical coupling constant β_C [3, 4]. If one fixes a number of lattice sites in the time direction N_τ and changes a number of sites in spatial directions N_σ, then the curves of the dependencies $g_4(\beta)$ will intersect each other in one point [8, 9]. The value of β in this point is a critical value for the lattice with $N_\tau = const$ and $N_\sigma \to \infty$. As shown above, the result of MC calculations of g_4 is the set of points; therefore one should fit data to find an intersection point of the cumulants. To locate this intersection point we use data from Tab. 3. The detailed procedure of the calculation of intersection point is described below. Values of the β_C received for various lattices are gathered in Tab. 4. For example, the following values of β_C was calculated in Ref. [3]: $1.8800(30)_{N_\tau=2}$, $2.2986(6)_{N_\tau=4}$, $2.4265(30)_{N_\tau=6}$, $2.5115(40)_{N_\tau=8}$; the next values was calculated in Ref. [4]: $1.87380(3)_{N_\tau=2}$, $2.29850(6)_{N_\tau=4}$, $2.51098(58)_{N_\tau=8}$. Listed values of β_C are in good agreement with our data (Tab. 4).

Let us consider the properties of the curve (3). First, as it seen from Tab. 2, the parameters of the curve based on the 600 data points, are merely the same as parameters of the curve based on the 25 data points. It leads to an important consequence: to estimate the parameters of a curve there is no need to perform the long MC calculation. Second, the parameter β_0 coincides (to within 2 up to 3 digits) with an inverse critical coupling constant β_C for a corresponding lattice. Using combination of both properties it is possible to estimate quickly a value of β_C on a lattice with any geometry $N_\tau \times N_\sigma^3$.

It is often necessary to construct and analyze a quantity which depends on Binder cumulants $g_4(\beta, N_\tau, N_\sigma)$ which is calculated using different N_τ and N_σ. Beta-function [10] is an example of such quantity. For lattices with identical values of N_τ the Binder cumulants should intersect in one point [8, 9] and beta-functions should self-intersect in a corresponding point. From Fig. 1, any three of curves do not cross in one point. Moreover, the position and the shape of curves are random variables which depend on a choice of a fitting interval, and also depend on the data amount. The interest causes studying of beta-function in the critical area. In this area the beta-function distorts mostly. This distortion appears due to many points of the cumulants intersections. In practice, applying fitting procedure to the sets of data from different lattices, one won't receive the set of cumulants, crossed in a point.

One needs to update the fitting procedure in such manner that required cumulants cross in one point. This is very similar to a problem of calculation of inverse critical coupling on a lattice. Considered problem is not trivial because of the condition, which is imposed on the curves during fitting.

It is possible to simplify fitting procedure of cumulants g_4, making changes into initial objective: we will search only for a point of cumulants crossing. The easiest way to demonstrate the given approach is to fit data by straight lines. As it is known, Binder cumulants are linear near critical region [8, 9]. We compute such lines for the data from critical regions of different lattices. Fitting results are in Tab.3. From this, we calculate coordinates of all possible points of crossing of the straight lines and calculate mean deviations for such coordinates. Received coordinates are the random values of the fitting interval and lattice data. And next, we find a point of the intersection of the cumulants as weighted average of coordinates considered (Tab. 4).

Table 3

Fitting of Binder cumulants by straight lines $a\beta + b$

Lattice	Parameters			Number of points	Fitting range	
	χ^2	a	b		$\beta_{\min}$	$\beta_{\max}$
$N_\tau = 2, N_\sigma = 8$	0.007	-14	25	100	1.875	1.885
$N_\tau = 2, N_\sigma = 12$	0.013	-30	56	100	1.875	1.885
$N_\tau = 2, N_\sigma = 16$	0.016	-33	61	100	1.875	1.885
$N_\tau = 2, N_\sigma = 20$	0.018	-43	79	100	1.875	1.885
$N_\tau = 2, N_\sigma = 24$	0.019	-53	98	81	1.875	1.883
$N_\tau = 2, N_\sigma = 28$	0.014	-39	71	61	1.875	1.881
$N_\tau = 2, N_\sigma = 32$	0.012	-33	61	71	1.875	1.882
$N_\tau = 4, N_\sigma = 8$	0.0087	-13	29	100	2.295	2.305
$N_\tau = 4, N_\sigma = 12$	0.035	-21	47	207	2.295	2.35
$N_\tau = 4, N_\sigma = 16$	0.038	-25	55	191	2.295	2.33
$N_\tau = 4, N_\sigma = 20$	0.083	-32	71	191	2.295	2.32
$N_\tau = 4, N_\sigma = 24$	0.13	-72	165	100	2.295	2.305
$N_\tau = 4, N_\sigma = 28$	0.11	-48.7	110.6	541	2.295	2.314
$N_\tau = 4, N_\sigma = 32$	0.16	-49	111.3	541	2.2997	2.314
$N_\tau = 4, N_\sigma = 36$	0.2	-65	147	600	2.28	2.30995
$N_\tau = 6, N_\sigma = 8$	0.012	-4.1	8.7	101	2.422	2.432
$N_\tau = 6, N_\sigma = 12$	0.023	-4.05	8.4	101	2.422	2.432
$N_\tau = 6, N_\sigma = 16$	0.035	-15	35	101	2.422	2.432
$N_\tau = 6, N_\sigma = 20$	0.11	-47	113	101	2.422	2.432
$N_\tau = 6, N_\sigma = 24$	0.17	-48	116	101	2.422	2.432
$N_\tau = 6, N_\sigma = 28$	0.25	-44	106	101	2.422	2.432
$N_\tau = 8, N_\sigma = 12$	0.024	-6.5	15	100	2.507	2.5169
$N_\tau = 8, N_\sigma = 16$	0.061	-11	26	100	2.507	2.5169
$N_\tau = 8, N_\sigma = 20$	0.071	-20	48	100	2.507	2.5169
$N_\tau = 8, N_\sigma = 24$	0.16	-35	87	100	2.507	2.5169

Table 4

Critical values of the inverse coupling constant and values of Binder cumulant in the critical point

N_τ	2	4	6	8
β_C	1.875	2.301	2.422	2.508
$g_4(\beta_C)$	-1.54	-1.5	-1.23	-1.27

It is easy to change fitting procedure using Tab. 4, so that appropriate cumulants will cross in one point. We impose the condition on the Eq. (3) that this curve should pass through the point with fixed coordinates (Tab. 4). Thus, one parameter of the curve (3) is excluded. We choose to exclude A_2, so the modified equation (3) for lattices with $N_\tau = 4$ looks:

$$g_4(\beta) = A_1 - \frac{(A_1 + 1.40864) \times (1 + 10^{(\beta_0 - 2.30086) \times p})}{1 + 10^{(\beta_0 - \beta) \times p}}. \qquad (4)$$

The fitting procedure of data by the modified curves (3) leads to convergence yet not always. The convergence depends on the excluded parameter and on the software chosen for fitting. It can be complicated to adopt this software for any given relation. Therefore, it is more usual to exclude some another parameter of (3).

Conclusions

Our calculations became possible due to technology of GPU computations. It is necessary to notice that usage of GPU during lattice calculations makes possible to gather a huge amount of the statistical data that allows studying of the Binder cumulants for a great number of various lattices.

We have performed high-statistics calculations of the Binder cumulant in SU(2) lattice gluodynamics. It is important that the gathered statistics allow us to construct and analyze the Binder cumulants. Based on such analysis we propose the function for fitting of cumulants. A few points of data (10 up to 20) are sufficient for computation of this function. The remarkable feature of the function offered is ability to estimate quickly the critical value of the inverse coupling constant β_C. The values of β_C calculated are in good agreement with the values known in the literature [3, 4].

The procedure of determination of the intersection point of the Binder cumulants, which are computed on lattices with different numbers of spatial sites and equal numbers of time sites, is considered. The updated fitting procedure for determination of cumulants, which are crossed in one point, is proposed.

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