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RADIAL MOTIONS IN THE GRAVITATIONAL FIELD OF A MASSIVE OBJECT WITH SCALAR AND ELECTRIC CHARGES

An exact solution of Einstein equations for a static spherically symmetric space-time that describes the scalar, electrostatic and gravitational fields of the central object with mass M, and scalar G and electric Q charges in the quasi global coordinates with $g_{00}\, g_{11} = -1$ is obtained. The radial motion of neutral test particles in this space is investigated. It is found that in the case of super-extremal electric charge in which $Q^2 > \gamma M^2$ the metric component g_{00}, playing the role of an effective potential in the equation of motion, has a minimum. This indicates on the existence of the hovering point that is a position of the stable static equilibrium of a test particle for some set of particle parameter values. The small deviations of the particle parameters from the equilibrium state will generate the harmonic oscillations of the particle near the hovering point. As an example, the frequency of the oscillations due to the small deviation of the particle energy is calculated. This result may serve as the classical approximation to the quantum-mechanical problem of normal modes for a massive object with electric charge and dynamical scalar field.

Key words: radial motion, scalar field, spherically-symmetric metric, hovering point.

Одержано точний розв'язок рівнянь Ейнштейна для статичного сферично-симетричного простору-часу, що описує скалярне, електростатичне та гравітаційне поля центрального об'єкта з масою M, скалярним G і електричним Q зарядами в квазіглобальних координатах, коли $g_{00}g_{11} = -1$. Досліджено радіальні рухи нейтральних пробних частинок у цьому просторі. Встановлено, що у випадку суперекстремального електричного заряду, коли $Q^2 > \gamma M^2$, компонента метрики g_{00}, що грає роль ефективного потенціалу в рівнянні руху, має мінімум. Це вказує на існування точки зависання, тобто положення стійкої статичної рівноваги пробної частинки для деякої сукупності значень параметрів частинки. Малі відхилення параметрів частинки від рівноважного стану породжують гармонічні коливання частинки поблизу знайденої точки зависання. Як приклад розраховано частоту коливань унаслідок малого відхилення енергії частинки. Цей результат може слугувати класичним наближенням до квантово-механічної задачі про нормальні моди для масивного об'єкта з електричним зарядом і з динамічним скалярним полем.

Ключові слова: радіальний рух, скалярне поле, сферично-симетрична метрика, точка зависання.

Получено точное решение уравнений Эйнштейна для статического сферически-симметричного пространства-времени, описывающего скалярное, электростатическое и гравитационное поля центрального объекта с массой M, скалярным G и электрическим Q зарядами в квазиглобальных координатах, когда $g_{00}\, g_{11} = -1$. Исследованы радиальные движения нейтральных пробных частиц в этом пространстве. Установлено, что в случае суперэкстремального электрического заряда, когда $Q^2 > \gamma M^2$, компонента метрики g_{00}, играющая роль эффективного потенциала в уравнении движения, имеет минимум. Это указывает на существование точки зависания, то есть положения устойчивого статического равновесия пробной частицы для некоторого набора значений параметров частицы. Малые отклонения параметров частицы от равновесного состояния порождают гармонические колебания частицы вблизи найденной точки зависания. В качестве примера рассчитана частота колебаний вследствие малого отклонения энергии частицы. Этот результат может служить классическим приближением для квантово-механической задачи о нормальных модах скалярного поля для массивного объекта с электрическим зарядом и динамическим скалярным полем.

Ключевые слова: радиальное движение, скалярное поле, сферически-симметричная метрика, точка зависания.

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1. Introduction

Probing gravitational field of compact massive objects plays an important role in General Relativity. One of the properties of interest is the hovering point that is an equilibrium position of test particles moving radially near the object. It has been shown [1] that such hovering points for charged and neutral test particles do exist for the Reissner-Nordström geometry in the case when it corresponds to a super-extremal value of electric charge.

The goal of this work is to study the impact of scalar charge on the above result. Although the gravitational field of a massive object with scalar and electric charges is considered in [2], the solution is obtained there in a sophisticated parametric form. That is why the present study starts with generalizing the more transparent solution with scalar charge and sub-extremal electric charge [3] to arbitrary values of electric charge.

2. General solution for metric

Let us consider the static spherically symmetric geometry and adopt the following Reissner-Nordström-like ansatz for the metric [3]

$$ds^2 = \frac{(z-a)(z-b)}{r^2} c^2 dT^2 - \frac{r^2}{(z-a)(z-b)} dz^2 - r^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \qquad (1)$$

with $r = r(z)$ and c being the velocity of light. Upon substituting $r^2 = (z-a)(z-b)e^{\lambda}$, one gets the component of the Einstein tensor

$$G_1^1 = e^{-\lambda}\left(\frac{\lambda'^2}{4} - \frac{(a-b)^2}{4(z-a)^2(z-b)^2}\right) \qquad (2)$$

where the prime denotes the derivative with respect to z.

Let the Einstein equations $G_\nu^\mu = 8\pi\gamma({}_{el}T_\nu^\mu + {}_{sc}T_\nu^\mu)/c^4$ contain the energy-momentum tensors of an electromagnetic field and a massless scalar field. For their static spherically symmetric configurations, the relations ${}_{el}T_0^0 = {}_{el}T_1^1 = -{}_{el}T_2^2 = -{}_{el}T_3^3$, ${}_{sc}T_0^0 = -{}_{sc}T_1^1 = {}_{sc}T_2^2 = {}_{sc}T_3^3$ hold, so that one can integrate the conservation-law equation $({}_{el}T_\nu^\mu + {}_{sc}T_\nu^\mu)_{;\mu} = 0$ to obtain

$${}_{el}T_1^1 = \frac{Q^2}{8\pi r^4}, \qquad {}_{sc}T_1^1 = -\frac{G^2 e^{\lambda}}{8\pi r^4} \qquad (3)$$

where integration constants Q and G are electric charge and scalar charge respectively.

The substitution of (2) and (3) into the Einstein equation for G_1^1 yields

$$\frac{\lambda'^2}{4}(z-a)^2(z-b)^2 - \frac{\gamma Q^2}{c^4} e^{-\lambda} - A^2 = 0 \qquad (4)$$

with $A^2 = (a-b)^2/4 - \gamma G^2/c^4$. The general solution to (4) in terms of $r(z)$ is [3]

$$r = \frac{1}{2A}[(z-a)(z-b)]^{(1-\alpha)/2}\left[C(z-b)^{\alpha} - \frac{\gamma Q^2}{Cc^4}(z-a)^{\alpha}\right] \qquad (5)$$

where

$$\begin{matrix} a \\ b \end{matrix} = \frac{\gamma M \pm \sqrt{\gamma^2 M^2 - \gamma(Q^2 - G^2)}}{c^2}, \quad C = \frac{\gamma M + \sqrt{\gamma^2 M^2 - \gamma Q^2}}{c^2}, \; \alpha = \frac{2A}{a-b} \qquad (6)$$

and M is the mass. However, it is out of scope of [3] that this solution splits into three different cases depending on whether constants under the square root signs in (6) are positive or not. The electrostatic potential A_0 and the scalar field φ, which obey

$$A_0' = -\frac{Q}{r^2}, \qquad \varphi' = -\frac{Ge^{\lambda}}{r^2} = -\frac{G}{(z-a)(z-b)}, \tag{7}$$

may have qualitatively different configurations in these three cases as well.

3. Metric and fields in particular cases

Let us examine separately the particular cases of the general solution.

1. First let $Q^2 > \gamma M^2 + G^2$ that implies that the electric charge is super-extremal $Q^2 > \gamma M^2$. In this case a and b are complex numbers according to (6). Then the scalar field derivative (7) is integrated for real z to produce

$$\varphi = G\,\mathrm{arc\,cot}\frac{c^2 z - \gamma M^2}{\sqrt{\gamma(Q^2 - G^2) - \gamma^2 M^2}}. \tag{8}$$

Applying de Moivre's formula to calculate the powers of $(z-a)$ and its complex conjugated $(z-b)$ in (5) and taking into account the relation $\varphi = G\arg(z-b)$, one gets

$$r = \sqrt{z^2 - 2\gamma Mz/c^2 + \gamma(Q^2 - G^2)/c^4}\left[\cos(\alpha\varphi/G) + \frac{\gamma M\sin(\alpha\varphi/G)}{\sqrt{\gamma Q^2 - \gamma^2 M^2}}\right] \tag{9}$$

where $\alpha = \sqrt{(Q^2 - \gamma M^2)/(Q^2 - \gamma M^2 - G^2)}$. This $r(z)$ dependence is monotonous and formulae (8) and (9) are well-defined when $z \ge z_0$, with z_0 denoting the largest root of $r(z) = 0$. Note that z is related to coordinate x of [2] by $x = \varphi(z)/(G\sqrt{Q^2 - \gamma M^2 - G^2})$.

It can be shown that in the absence of the scalar field, one has $r(z) = z$ and metric (1) reduces to that of the naked Reissner-Nordström singularity having no horizons. If the scalar field is present, the metric coefficient $g_{00} = (z-a)(z-b)/r^2$ does not vanish anywhere on the real z axis because a and b are complex. Consequently, in this case there are no horizons and the metric has a naked singularity at the origin as well. In Fig. 1 the plots of g_{00} (left) and the scalar field φ (right) versus r calculated according to (1), (8) and (9) are depicted by the solid lines. The calculation was made for $M = 1$, $Q = 1.1$, $M = 1$ and $G = 0.1$ in units in which $\gamma = c = 1$.

2. Now let $\gamma M^2 < Q^2 < \gamma M^2 + G^2$, so that the electric charge is super-extremal and bounded from above. The main difference from the first case is that now α becomes imaginary. As a consequence, the metric coefficients in (1) are defined through

$$r = \sqrt{z^2 - 2\gamma Mz/c^2 + \gamma(Q^2 - G^2)/c^4}\left[\cos(\hat{\alpha}\hat{\varphi}/G) + \frac{\gamma M\sin(\hat{\alpha}\hat{\varphi}/G)}{\sqrt{\gamma Q^2 - \gamma^2 M^2}}\right] \tag{10}$$

that differs from (9) only in the replacement $\alpha \Rightarrow \hat{\alpha} = \alpha/i$, $\varphi \Rightarrow \hat{\varphi}$ where

$$\hat{\alpha} = \sqrt{\frac{Q^2 - \gamma M^2}{\gamma M^2 - Q^2 + G^2}}, \quad \hat{\varphi} = G\,\mathrm{arc\,coth}\frac{c^2 z - \gamma M^2}{\sqrt{\gamma^2 M^2 - \gamma(Q^2 - G^2)}} \tag{11}$$

and $\hat{\varphi}$ designates the scalar field for this case. Note that the so-obtained $\hat{\varphi}$ satisfies (7). Once again g_{00} has no zeros for $z \ge z_0$ and thus there is a naked singularity. For this case, the dependences of g_{00} and φ on r are depicted by the dotted lines in Fig. 1. The calculation was made using the values $M = 1$, $Q = 1.1$, $M = 1$ and $G = 0.5$.

3. Let $Q^2 < \gamma M^2$ that refers to the solution with sub-extremal electric charge, which was analyzed in [3]. In this case all the constants in (6) are real and hence the initial $r(z)$ dependence (5) is well-defined and determines the metric via (1). Since $r(z)$ vanishes at a and b ($a > b$), z is now restricted to $z \geq a$. In the absence of the scalar field ($G = 0$), one has $r(z) = z$ and $z = a$ is the event horizon of the Reissner-Nordström black hole. However, if $G \neq 0$, there are no horizons and $z = a$ turns out to be a naked singularity [3]. The corresponding g_{00} and φ are depicted by the dashed lines in Fig. 1. These plots were obtained using the values $M = 1$, $Q = 0.9$, $M = 1$ and $G = 0.5$.

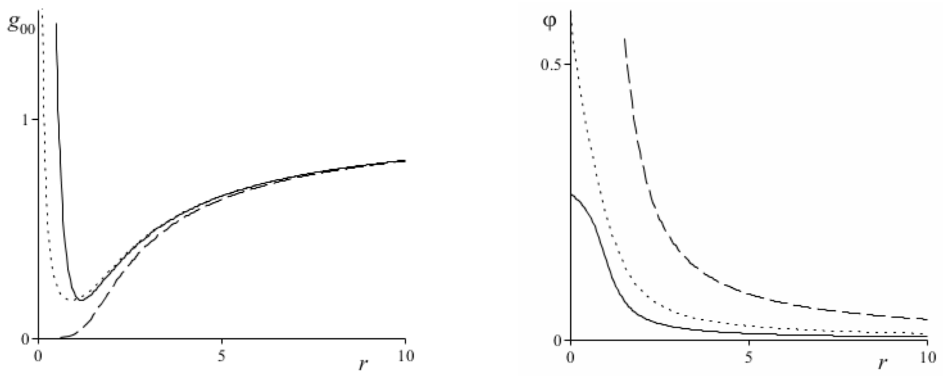

Fig. 1. Typical plots of g_{00}(r) (left) and φ(r) (right) for $Q^2 > \gamma M^2 + G^2$ (solid line), $\gamma M^2 < Q^2 < \gamma M^2 + G^2$ (dotted line) and $Q^2 < \gamma M^2$ (dashed line).

It is seen from Fig. 1 that the metric is asymptotically flat in each of the cases though its behavior at small r varies depending on the value of electric charge. So does the scalar field, which is finite at origin only in the case of super-extremal electric charge. In its turn, the electrostatic potential behaves approximately as $A_0 \approx Q / r$ in all the three cases because it always obeys (7) where $r(z)$ is almost linear in z especially for large z.

As a remark, it should be added that the system under consideration has also two limiting cases $\gamma M^2 = Q^2$ and $Q^2 = \gamma M^2 + G^2$, which were not considered here for the sake of brevity. The corresponding metrics can be found in [3].

4. Radial motions

Consider a neutral test particle with mass m and energy E. Its radial motion in the space-time with metric (1) is described by the world-line equation

$$\left(mc^2 \frac{dz}{ds} \right)^2 = \frac{1}{g_{11} g_{00}} \left(m^2 c^4 g_{00} - E^2 \right) = E^2 - m^2 c^4 e^{-\lambda} \tag{12}$$

in which the component g_{00} plays the role of an effective potential. Since the metric is shown to be asymptotically flat, $g_{00} = e^{-\lambda} \to 1$ as $r \to \infty$. Then from (12) it follows that the motion is finite (infinite) for particles with $E < mc^2$ ($E \geq mc^2$).

If the electric charge is sub-extremal $Q^2 < \gamma M^2$ (case 3), from Fig. 1 one sees that $g_{00} = e^{-\lambda} \to 0$ as $r \to 0$. Hence, $(mc^2 dz / ds)^2 \to E^2 > 0$ and the particle moving inwards will fall into the singularity. If the electric charge is super-extremal (cases 1 and 2), $(mc^2 dz / ds)^2 \to -\infty$ as $r \to 0$, so that the particle cannot reach the singularity. In these cases there exists a hovering point of the test particle. According to [4], it is located at the minimum of $g_{00} = e^{-\lambda}$. For case 1, the corresponding areal radius is

$$r_{eq}^2 = \frac{\gamma Q^2}{c^4} \frac{Q^2 - \gamma M^2 - G^2}{(Q^2 - \gamma M^2)\sin^2\left(\sqrt{\frac{Q^2 - \gamma M^2}{Q^2 - \gamma M^2 - G^2}} \arcsin\sqrt{\frac{\gamma M^2}{Q^2}}\right)} \quad (13)$$

that reduces to the Reissner-Nordström value $r_{RN-eq} = Q^2/(Mc^2)$ [1] when $G = 0$. For case 2, one should invert the signs in $Q^2 - \gamma M^2 - G^2$ and replace sin by sinh in (13).

Suppose the particle energy is slightly greater than that of the particle resting at hovering point. Then the small radial oscillations of the particle near the hovering point will occur with the frequency calculated with respect to the time of a remote observer [5]

$$\Omega = \left(\frac{\partial^2 H}{\partial z^2}\frac{\partial^2 H}{\partial p^2}\right)^{1/2}_{r_{eq}=0,\ p=0}. \quad (14)$$

In the case of $Q^2 > \gamma M^2 + G^2$ (case 1) the last formula yields

$$\Omega = \frac{c^3}{\sqrt{\gamma}} \frac{(Q^2 - \gamma M^2)^{3/2}}{Q^2(Q^2 - \gamma M^2 - G^2)} \sin^2\left(\sqrt{\frac{Q^2 - \gamma M^2 - G^2}{Q^2 - \gamma M^2}} \arcsin\frac{\sqrt{\gamma}M}{\sqrt{Q^2}}\right) \quad (15)$$

that in the absence of the scalar charge ($G = 0$) reduces to the Reissner-Nordström result [5]. The frequency for the case of $\gamma M^2 < Q^2 < \gamma M^2 + G^2$ (case 2) is obtained from (14) by the same replacements as described after equation (13).

5. Conclusions

In this work the solution to the Einstein equations for the massive object with arbitrary values of scalar and electric charges has been presented. It is shown that the radial motions of neutral test particles in this geometry depend on the electric charge of the object. If and only if this charge attains a super-extremal value, there is a stable equilibrium position of the test particle. It is interesting to explore the implications of this effect at the quantum level since it may give rise to the scalar field normal modes.

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