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A PROBLEM OF COSMOLOGICAL TIME IN THE DE SITTER SPACE-TIME

In the paper a new representation of de Sitter solutions is obtained by a transformation from coordinates of curvatures to synchronous coordinate systems. The special cases of this representation are those by S. Hawking, G. Ellis and E. Shrödinger. Obtained generalized representation study is performed by the method of embedding into the 5-dimensional flat space. The fundamental formulae for embeddings are derived and explanation of the obtained results is given. The physical sense of integration functions is ascertained. The feasibility of the matching of de Sitter solution and the Tolman world is shown. The de Sitter world may be an initial state for the Tolman Universe because the choice of integration functions is a choice of a synchronous coordinate system, in which it is possible to match Tolman and de Sitter metrics. Since the generalized representation of de Sitter solution does not constrain the values of integration functions, there are infinitely many such synchronous coordinate systems and, therefore, the matching of de Sitter solutions and general Tolman solution is possible. So, it is impossible to introduce a notion of universal cosmological time in the de Sitter Universe.

Keywords: Tolman and de Sitter metrics, synchronous coordinate system, 5-dimensional flat space, cosmological time.

У роботі методом переходу від координат кривин до синхронних систем координат отримано нове представлення розв'язків де Сіттера, окремими випадками якого є представлення, отримані С. Хокінгом, Дж. Еллісом й Е. Шрьодінґером. Дослідження отриманого узагальненого представлення проводилось методом вкладання в п'ятивимірний плоский простір. Виведено основні формули вкладань і дано інтерпретацію отриманих результатів. Установлено фізичний сенс функцій інтегрування. Показано можливість зшиття розв'язку де Сіттера зі світом Толмена. Світ де Сіттера може бути початковим станом для Всесвіту Толмена з огляду на те, що вибір функцій інтегрування становить собою вибір синхронної системи координат, у якій можливе зшиття метрик Толмена й де Сіттера. Оскільки узагальнене представлення розв'язку де Сіттера не накладає обмежень на значення функцій інтегрування, то синхронних систем координат існує безліч і, отже, можливим є зшиття розв'язків де Сіттера з загальним розв'язком Толмена. Тому поняття універсального космологічного часу у світі де Сіттера запровадити неможливо.

Ключові слова: метрики Толмена й де Сіттера, синхронна система координат, п'ятивимірний плоский простір, космологічний час.

В работе методом перехода от координат кривизн к синхронным системам координат получено новое представление решений де Ситтера, частными случаями которого являются представления, полученные С. Хокингом, Дж. Эллисом и Э. Шредингером. Исследование полученного обобщенного представления проводилось методом вложения в пятимерное плоское пространство. Выведены основные формулы вложений и дана интерпретация полученным результатам. Установлен физический смысл функций интегрирования. Показана возможность сшивки решения де Ситтера с миром Толмена. Мир де Ситтера может быть начальным состоянием для Вселенной Толмена ввиду того, что выбор функций интегрирования представляет собой выбор синхронной системы координат, в которой возможна сшивка метрик Толмена и де Ситтера. Поскольку обобщенное представление решения де Ситтера не накладывает ограничений на значения функций интегрирования, то синхронных систем координат имеется бесконечное множество и, следовательно, возможна сшивка решений де Ситтера с общим решением Толмена. Поэтому понятие универсального космологического времени в мире де Ситтера ввести невозможно.

Ключевые слова: метрики Толмена и де Ситтера, синхронная система координат, пятимерное плоское пространство, космологическое время.

1. Introduction

As in any other theory, in General Relativity there are many issues under discussion. In view of complexity of field equations obtaining exact solutions becomes feasible only in spaces of high dimensionality. Therefore, every exact solution is considerably preferred than any kind of approximations. The cosmological Big Bang model is based on Friedmann exact solutions. But they imply some difficulties including the initial singularity problem. One of the ways of its suppression is matching of the Friedmann world and an exact solution of field equations without initial singularity. Among the inflationary Universe models such solution is de Sitter one describing initial period of the Universe.

Modern observations indicate inhomogeneous distribution of matter [1]. Thus, to describe the Universe we use Tolman solution, the exact solution of field equations with equation of state $p = 0$. Upon solving such problem there arises the difficulty of constructing the cosmological model where the current stage of the Universe evolution is described by the Tolman solution and initial stage – by that of de Sitter.

A. A. Friedmann proved that the matter in the Universe cannot be quiescent. The Universe cannot be stationary. It can either expand or collapse. So there was theoretically revealed the necessity of overall evolution of the Universe. In line with this theory, the metric of the homogeneous and isotropic Universe with equation of state $p = 0$ can be written as

$$ds^2 = dt^2 - a^2(t)\left[dr^2/\left(1-kr^2\right) + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right], \tag{1}$$

where $k = +1, -1, 0$ for the closed, open, and flat Friedmann Universe, correspondingly; $a(t)$ is the Universe “radius”, to be more precise, its scale factor (the full size of the Universe can be infinite). The flat Universe corresponds to the metric

$$ds^2 = dt^2 - a^2(t)\left(dx^2 + dy^2 + dz^2\right). \tag{2}$$

At any point of time its spatial part describes an ordinary 3-dimensional Euclidean space and when $a(t)$ is constant, it describes the Minkowski space. The closed, open, and flat Friedmann solutions describing a homogeneous and isotropic distribution of dust-like matter with equation of state $p = 0$ can be written in synchronous coordinate systems [2]:

$$ds^2 = d\tau^2 - a_0^2 \sin^4(\eta/2)\left(dR^2 + \sin^2 R d\sigma^2\right), \quad \tau = a_0(\eta - \sin\eta)/2\,, \tag{3}$$

$$ds^2 = d\tau^2 - a_0^2 \mathrm{sh}^4(\eta/2)\left(dR^2 + \mathrm{sh}^2 R d\sigma^2\right), \quad \tau = a_0(\mathrm{sh}\,\eta - \eta)/2\,, \tag{4}$$

$$ds^2 = d\tau^2 - (\tau/b)^{4/3}\left(dR^2 + R^2 d\sigma^2\right), \quad b = (2/3)a_0^{-1/2} \tag{5}$$

where $a_0 = 4M\gamma/\left(3\pi c^2\right)$. So, every Friedmann model class corresponds to a definite time.

Regardless of the model class the scale factor of the Universe vanishes at some time $t = 0$, and matter density at this time becomes infinite. At once tensor of space curvature goes to infinity also. This is a reason to call $t = 0$ a point of initial cosmological singularity. So, every Friedmann model implies a singularity where all nature laws fail.

Thus, when formulating cosmological models of the Universe one faces the problem of infinite density of energy at zero time. But there is a way to avoid it. It consists in using a solution of Einstein equations without initial singularity for the Universe description at the initial stage of evolution. One of those solutions is de Sitter metric, and a cosmological model is made by the matching of de Sitter metric and the Friedmann one.

2. The generalized representation of de Sitter solution

The metric of de Sitter, written in coordinate system of curvatures, has the form:

$$ds^2 = \left(1 - r^2 / a^2\right) dt^2 - \left(1 - r^2 / a^2\right)^{-1} dr^2 - r^2 d\sigma^2 \tag{6}$$

where $a = (8/3) c^{-4} \pi \gamma \varepsilon$; γ is Newton constant; ε is energy density; c is speed of light, further we assume $c = 1$. Considering equation of state $\varepsilon + p = 0$, we conclude p to be negative as well [3], but none of existing kinds of matter can produce such negative pressure, which comes up to density in magnitude. So, de Sitter Universe must be empty.

A general solution of cosmological form in synchronous coordinate systems is

$$ds^2 = d\tau^2 - T^2(\tau)\left(dR^2 + \psi^2(R) d\sigma^2\right) \tag{7}$$

where cases $\psi(R) = \sin R$, $\operatorname{sh} R$ and R are possible. In synchronous systems coordinate time coincides with true time, time lines are geodesics normal to hypersurface $\tau = \text{const}$.

In Ref. [4] a general method of transformation between the metrics

$$ds^2 = A(r) dt^2 - B(r) dr^2 - r^2 d\sigma^2 , \tag{8}$$

$$ds^2 = d\tau^2 - \exp[\lambda(R,\tau)] - r^2(R,\tau) d\sigma^2 \tag{9}$$

is given; the method comes to the integration of combined equations relating curvature and synchronous coordinates:

$$\begin{gathered} (dr/d\tau)^2 = f^2(R) - 1 + r^2(R,\tau)/a^2, \; (dr/d\tau) = f(R)/\left(1 - r^2/a^2\right), \\ \exp[\lambda(R,\tau)] = f^{-2}(R)(dr/dR)^2. \end{gathered} \tag{10}$$

Here $f(R)$ makes sense the total energy of a particle. Depending on $f(R)$ with assumption of homogeneity and isotropy we obtain three types of synchronous coordinates:

$$ds^2 = d\tau_1^2 - a^2 \operatorname{sh}(\tau_1/a)\left(dR^2 + \operatorname{sh}^2 R d\sigma^2\right), \quad f^2(R) > 1, \tag{11}$$

$$ds^2 = d\tau_2^2 - a^2 \operatorname{ch}(\tau_2/a)\left(dR^2 + \sin^2 R d\sigma^2\right), \quad f^2(R) < 1, \tag{12}$$

$$ds^2 = d\tau_3^2 - a^2 \exp(2\tau_3/a)\left(dR^2 + R^2 d\sigma^2\right), \quad f^2(R) = 1. \tag{13}$$

The metric (11) defines the hyperbolic type of motion and the world with a constant negative space curvature, obtained by Lemaître and Robertson [5]; Eq. (12) describes the elliptic motion type, the space curvature of the world is positive, obtained by Hawking and Ellis [6]; Eq. (13) determines the parabolic type and the world with zero space curvature [7].

Thus, only three solutions of cosmological type describing the curved space can be formulated in terms of coordinates of curvatures. All of them describe the same metric, which is de Sitter one, and the only solution, which can be expressed in a cosmological form, given by Eq. (6). If the same metric corresponds to two different solutions of cosmological type, it always can be described also in curvature coordinates (8) where all the metric coefficients depend on radial coordinate. Then the equation of state is $\varepsilon + p = 0$. It can be either de Sitter metric or empty space (as a special case of de Sitter solution $\varepsilon = p = 0$).

The de Sitter Universe can be described by three different solutions (11)-(13). For every such solution one may introduce the universal time. So, the de Sitter world proves to have three such times. The analysis of all three models shows, that only the coordinate system (12) is complete, the time in this system is varying from $-\infty$ to ∞. But it might be wrong to prefer this coordinate system and consider time τ_2 as cosmological time. If one considers a phase transition between the worlds at some time $\tau = \text{const}$, then it is necessary to choose τ_1, τ_2 or τ_3 as universal time, depending on the Friedmann Universe we study. So, in the de Sitter world there is no universal time. It appears only in the transformation to the Friedmann Universe or any other Friedmann-like world (7).

Current knowledge of 3-dimensional distribution of galaxies gives grounds to consider that our Universe is not homogeneous and isotropic [1]. So, to describe it instead of Friedmann models one should use more general Tolman models [5] describing inhomogeneous dust cloud with equation of state $p = 0$ and $\varepsilon = \varepsilon(R, \tau)$.

An arbitrary Tolman solution can be written in a form

$$ds^2 = d\tau^2 - (dr/dR)^2 dR^2 / f^2(R) - r^2(R,\tau) d\sigma^2, \quad d\sigma^2 = d\theta^2 + \sin^2\theta d\varphi^2. \tag{14}$$

Here for the hyperbolic $f^2(R) > 1$, elliptic $f^2(R) < 1$ and parabolic $f^2(R) = 1$ types of solutions we have:

$$r = F(R)\,\text{sh}^2(\alpha/2)/\left(f^2(R) - 1\right), \quad \tau - \tau_0(R) = F(R)(\text{sh}\,\alpha - \alpha)/2\left[f^2(R) - 1\right]^{3/2}; \tag{15}$$

$$r = F(R)\sin^2(\alpha/2)/\left(1 - f^2(R)\right), \quad \tau - \tau_0(R) = F(R)(\alpha - \sin\alpha)/2\left[1 - f^2(R)\right]^{3/2}; \tag{16}$$

$$r = \left[3F^{1/2}(R)(\tau - \tau_0(R))/2\right]^{2/3}, \tag{17}$$

where $F(R)$, $f(R)$ and $\tau_0(R)$ are arbitrary integration functions ($F(R)$ is a total mass of substance inside the layer R; $f(R)$ defines geometry of 3-dimensional part of solution; $\tau_0(R)$ is the initial time, different for every layer).

Any cosmological model is made by the matching of de Sitter metric with that of Tolman on a hypersurface $\tau = \text{const}$. In Ref. [7] it was constructed the Tolman – de Sitter cosmological model where the standard representation of de Sitter solution (11)-(13) was used. It was shown that an arbitrary Tolman solution (11) cannot be matched with de Sitter world. It is possible only for a certain type of Tolman's models, particularly, for all types of Friedmann models. So, we face the problem of searching for more general representation of de Sitter solution to match it with an arbitrary Tolman's solution (11).

Integrating Eqs. (10) we introduce integration functions $\tau_0(R)$ and $X(R)$. The first of Eqs. (10) with $r = a\left(1 - f^2(R)\right)^{1/2}\text{ch}(\alpha/a)$ leads to $\alpha = \tau + \tau_0(R)$. $f(R) = \cos R$ gives

$$r = a\sin R\,\text{ch}\,\beta, \quad \beta \equiv (\tau + \tau_0(R))/a. \tag{18}$$

Integration of the second expression (10) with the same substitution gives:

$$t = a\left(\text{Arcth}[\cos R\,\text{cth}\,\beta] + X(R)\right). \tag{19}$$

The expressions (18) and (19) represent the relationship of curvature coordinates and synchronous ones for the elliptic type of motion. Using the third equation (10), we find:

$$\exp[\lambda(R,\tau)] = a^2\text{ch}^2\beta + \text{tg}^2 R\,\text{sh}^2\beta(d\tau_0/dR)^2 + a\,\text{tg}\,R\,\text{sh}\,2\beta(d\tau_0/dR). \tag{20}$$

Applying the foregoing calculations, we can write the metric of the elliptic de Sitter world as:

$$ds^2 = d\tau^2 - \left(a^2 \mathrm{ch}^2\beta + \mathrm{tg}^2 R\, \mathrm{sh}^2\beta (d\tau_0 / dR)^2 + a\, \mathrm{tg}\, R\, \mathrm{sh}\, 2\beta (d\tau_0 / dR)\right) dR^2 - \\ - a^2 \sin^2 R\, \mathrm{ch}^2\beta d\sigma^2. \tag{21}$$

Using this method, we can obtain similar results for another two types of de Sitter solution. Replacing $f(R) = \mathrm{ch}\, R$, metric of the hyperbolic de Sitter world takes the form:

$$ds^2 = d\tau^2 - \left(a^2 \mathrm{sh}^2\beta + \mathrm{th}^2 R\, \mathrm{ch}^2\beta (d\tau_0 / dR)^2 + a\, \mathrm{th}\, R\, \mathrm{sh}\, 2\beta (d\tau_0 / dR)\right) dR^2 - \\ - a^2 \mathrm{sh}^2 R\, \mathrm{sh}^2\beta d\sigma^2; \tag{22}$$

and when $f^2(R) = 1$, for the metric of the parabolic de Sitter world we have:

$$ds^2 = d\tau^2 - \exp(2\beta)\left[a^2 + R^2 (d\tau_0 / dR)^2 + 2aR(d\tau_0 / dR)\right] dR^2 - a^2 R^2 \exp(2\beta) d\sigma^2. \tag{23}$$

The metrics (21)-(23) determine a generalized representation of the de Sitter Universe. When arbitrary function $\tau_0(R)$ exists, any metric for every of three types of solutions is represented by infinite set of metrics, each characterized by defined value of $\tau_0(R)$ and, therefore, by defined time. For every such cosmological solution one can introduce universal time. So, in the de Sitter Universe it appears to be infinitely many those times, not only three, as considered recently. In other words, the de Sitter Universe can be split by infinite number of possibilities into the worlds, each characterized by the defined time value and one of three types of space curvature: positive, negative and zero.

3. Study of generalized representation by embedding into 5-dimensional flat space

One of the simplest ways to study geometric properties of the de Sitter Universe is method of embedding the 4-dimensional de Sitter world into the 5-dimensional flat space and considering the space-time geometry on the hypersurface, which is the de Sitter world embedded in five-dimensionality. This problem is discussed in Refs. [6, 8-10].

Embedding the de Sitter solutions (6) in R^5, which metric is

$$ds^2 = dW^2 - dV^2 - dX^2 - dY^2 - dZ^2, \tag{24}$$

we find by technique of undefined functions how (W, V, X, Y, Z) depend on (t, r, θ, φ):

$$W = \mathrm{sh}(t/a)\left[a^2 - r^2\right]^{1/2}, \quad V = \mathrm{ch}(t/a)\left[a^2 - r^2\right]^{1/2} \\ X = r\cos\theta, \quad Y = r\sin\theta\sin\varphi, \quad Z = r\sin\theta\sin\varphi. \tag{25}$$

The hypersurface, which is de Sitter model embedded in R^5, is given by

$$X^2 + Y^2 + Z^2 + V^2 - W^2 = a^2, \tag{26}$$

i.e. it is a 5-dimensional hyperboloid. To make object of our study more graphic let's consider the section $\theta = 0$ of the complete model (26). R^5-space transforms into R^3 with the metric

$$ds^2 = dW^2 - dV^2 - dX^2, \tag{27}$$

and the de Sitter Universe turns into one-lane equiaxed hyperboloid:

$$X^2 + V^2 - W^2 = a^2. \tag{28}$$

The hyperboloid coordinates defining the embedded elliptic de Sitter model are [6]:

$$W = a\,\mathrm{sh}(\tau_2/a), \quad V = a\,\mathrm{ch}(\tau_2/a)\cos R, \quad X = a\,\mathrm{ch}(\tau_2/a)\sin R\cos\theta,$$
$$Y = a\,\mathrm{ch}(\tau_2/a)\sin R\sin\theta\cos\varphi, \quad Z = a\,\mathrm{ch}(\tau_2/a)\sin R\sin\theta\sin\varphi. \qquad (29)$$

Time-like geodesics are hyperbolae converging monotonically to some spatial distance and then diverging to infinity again; space-like ones are S^3-spheres of constant positive curvature; when choosing W as a time, both particle and event horizons appear for a time-like observer because of past and future boundary space-likeness; τ_2, R, θ, φ cover all the space ($-\infty < \tau_2 < \infty$, $0 \le R \le \pi$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$), so system (12) is complete.

The hyperboloid coordinates for the embedded hyperbolic de Sitter model are [2]:

$$W = a\,\mathrm{sh}(\tau_1/a)\mathrm{ch}\,R, \quad V = a\,\mathrm{ch}(\tau_1/a), \quad X = a\,\mathrm{ch}(\tau_2/a)\sin R\cos\theta,$$
$$Y = a\,\mathrm{sh}(\tau_1/a)\mathrm{sh}\,R\sin\theta\cos\varphi, \quad Z = a\,\mathrm{sh}(\tau_1/a)\mathrm{sh}\,R\sin\theta\sin\varphi. \qquad (30)$$

The system (11) is incomplete as V cannot be less than a, it covers only a part of hyperboloid. In view of isotropy of past and future boundaries there are no particle and event horizons.

The hyperboloid coordinates for the embedded parabolic de Sitter world are [6]:

$$W = a\left[\mathrm{sh}(\tau_3/a) + R^2\exp(\tau_3/a)/2\right], \quad V = a\left[\mathrm{ch}(\tau_3/a) - R^2\exp(\tau_3/a)/2\right],$$
$$X = a\mathrm{R}\exp(\tau_3/a)\cos\theta, \; Y = a\mathrm{R}\exp(\tau_3/a)\sin\theta\cos\varphi, \; Z = a\mathrm{R}\exp(\tau_3/a)\sin\theta\sin\varphi. \qquad (31)$$

The coordinate system (13) covers only a half of hyperboloid in region $V + W > 0$. In the model for any time-like observer there are no particle and past and future event horizons.

The hyperboloid coordinates for the generalized elliptic model are of the form:

$$W = a\left[\cos R\,\mathrm{ch}\beta\,\mathrm{sh}\,X(R) + \mathrm{ch}\,X(R)\mathrm{sh}\,\beta\right],$$
$$V = a\left[\cos R\,\mathrm{ch}\beta\,\mathrm{ch}\,X(R) + \mathrm{sh}\,X(R)\mathrm{sh}\beta\right], \quad X = a\sin R\,\mathrm{ch}\beta \qquad (32)$$

For $X(R) = 0$ and $\tau_0(R) = 0$ we have the standard representation (29). When $\theta = 0$ Eq. (21) can be written as follows:

$$ds^2 = d\tau^2 - \left(a^2\mathrm{ch}^2\beta + \mathrm{tg}^2 R\,\mathrm{sh}^2\beta\left(d\tau_0/dR\right)^2 + a\,\mathrm{tg}\,R\,\mathrm{sh}\,2\beta\left(d\tau_0/dR\right)\right)dR^2 . \qquad (33)$$

Finding dW^2, dV^2, dX^2, making combination like $ds^2 = dW^2 - dV^2 - dX^2$ (to be the R^3-metric) and then comparing the $d\tau^2$ and dR^2 coefficients, we reveal that functions $\tau_0(R)$ and $X(R)$ for the elliptic type of solution are related as:

$$-\left(d\tau_0(R)/dR\right) = a\cos R\left(dX(R)/dR\right). \qquad (34)$$

By analogy, the hyperboloid coordinates for the generalized hyperbolic model are:

$$W = a\left[\mathrm{ch}\,R\,\mathrm{sh}\beta\,\mathrm{ch}\,X(R) + \mathrm{sh}\,X(R)\mathrm{ch}\beta\right],$$
$$V = a\left[\mathrm{ch}\,R\,\mathrm{sh}\beta\,\mathrm{sh}\,X(R) + \mathrm{ch}\,X(R)\mathrm{ch}\,\beta\right], \quad X = a\,\mathrm{sh}\,R\,\mathrm{sh}\beta, \qquad (35)$$

for $X(R) = 0$ and $\tau_0(R) = 0$ we have the standard Eq. (30). When $\theta = 0$, Eq. (22) gives

$$ds^2 = d\tau^2 - \left(a^2\mathrm{sh}^2\beta + \mathrm{th}^2 R\,\mathrm{ch}^2\beta\left(d\tau_0/dR\right)^2 + a\,\mathrm{th}\,R\,\mathrm{sh}\,2\beta\left(d\tau_0/dR\right)\right)dR^2, \qquad (36)$$

and functions $\tau_0(R)$ and $X(R)$ for the hyperbolic type of solution are related by

$$-\left(d\tau_0(R)/dR\right) = a\,\mathrm{ch}\,R\left(dX(R)/dR\right). \qquad (37)$$

The hyperboloid coordinates for the generalized parabolic model are:

$$W = a\left[\mathrm{ch}X(R)\left(\mathrm{sh}\,\beta + R^2\exp(\beta)/2\right) + \mathrm{ch}\,X(R)\left(\mathrm{ch}\,\beta - R^2\exp(\beta)/2\right)\right],$$
$$V = a\left[\mathrm{ch}\,X(R)\left(\mathrm{ch}\,\beta - R^2\exp(\beta)/2\right) + \mathrm{sh}\,X(R)\left(\mathrm{sh}\,\beta + R^2\exp(\beta)/2\right)\right], \quad X = a\mathrm{R}\exp(\beta). \qquad (38)$$

where for $X(R) = 0$ and $\tau_0(R) = 0$ we come to Eq. (31). When $\theta = 0$ Eq. (23) looks like

$$ds^2 = d\tau^2 - \exp(2\beta)\left(a^2 + R^2(d\tau_0/dR)^2 + 2aR(d\tau_0/dR)\right)dR^2, \qquad (39)$$

and functions $\tau_0(R)$ and $X(R)$ for the parabolic type of solution are related as:

$$-\left(d\tau_0(R)/dR\right) = a\left(dX(R)/dR\right). \qquad (40)$$

Eqs. (32), (35), and (38) are the formulae of embedding the generalized representation of de Sitter solutions in 5-dimensional flat space with additional conditions (34), (37) and (40). Note that pairs of $\tau_0(R)$ and $X(R)$ functions for each of three cases are different.

Let us ascertain the geometric sense of $X(R)$ and $\tau_0(R)$ for the elliptic model. The general Lorentz transform for coordinates (28), as the metric is pseudo-Euclidean, is

$$W = V'\mathrm{sh}\,\psi + W'\mathrm{ch}\,\psi, \quad W = V'\mathrm{sh}\,\psi + W'\mathrm{ch}\,\psi. \qquad (41)$$

Under Lorentz transformations the shape of our hyperboloid does not change, only curves made by section of the hyperboloid with certain surfaces, reshapes.

Difference of squares of coordinates (32) is

$$V^2 - W^2 = [\mathrm{ch}^2\beta\cos^2 R - sh^2\beta]a^2. \qquad (42)$$

It coincides with that of (29), then as ψ in this case we take $X(R)$. So, $X(R)$ is “angle of rotation” in the 5-dimensional flat space, and $\tau_0(R)$ is a function of angle of rotation. For us it is important to find time lines and space sections. Note that we can obtain in general only the time lines. There is no way to get in general the spatial sections without concretization of function $\tau_0(R)$ (one can represent them geometrically on hyperboloid).

The general form of the time lines $R = \mathrm{const}$ for the elliptic model is:

$$[\mathrm{ch}\,X(R)V - \mathrm{sh}(R)W]^2 - [\mathrm{ch}\,X(R)W - \mathrm{sh}\,X(R)V]^2\cos^2 R = a^2\cos^2 R. \qquad (43)$$

The space-like geodesics are clump of circles and ellipses each being the circle in synchronous coordinate system characterized by the defined value of $\tau_0(R)$ [8].

By analogy, we find the time lines for the hyperbolic and parabolic worlds:

$$\mathrm{ch}^2 R[\mathrm{ch}\,X(R)V - \mathrm{sh}\,X(R)W]^2 - [\mathrm{ch}\,X(R)W - \mathrm{sh}\,X(R)V]^2 = a^2\mathrm{ch}^2 R; \qquad (44)$$

$$[\mathrm{ch}\,X(R)V - \mathrm{sh}\,X(R)W]^2 - [\mathrm{ch}\,X(R)W - \mathrm{sh}(R)V]^2 = a^2\left(1 - R^2\right). \qquad (45)$$

Lines $\tau = \mathrm{const}$ for a hyperbolic model are hyperbolae and curves overlapping them upon rotation of coordinate system in R^3; similarly, for a parabolic model lines $\tau = \mathrm{const}$ represent parabolas etc. The straight lines are isotropic geodesics.

A peculiarity of the considered representation should be emphasized. Choosing a type of $\tau_0(R)$, we can obtain the equations for geodesics corresponding to another model within the model in study. Thus, $\tau_0(R) = \ln \sin R$ for the elliptic model gives the spatial sections

$$W + X = \exp(\tau) = \text{const.} \tag{46}$$

And this is nothing else but parabolas, which are time lines of a parabolic model. Hence, choosing the function $\tau_0(R)$, we can change the world type to another.

Returning to the construction of Tolman – de Sitter cosmological model, one concludes the de Sitter world can be the initial state for the Tolman Universe, because the choice of $\tau_0(R)$ is the choice of synchronous coordinates, in which it is possible to match the Tolman metric and that of de Sitter. As the generalized representation of de Sitter solution does not constrain the values of $\tau_0(R)$, there is an infinite set of synchronous coordinate systems and, therefore, the matching of de Sitter solutions and the general Tolman solution (14) is feasible.

4. Conclusions

In the paper a new representation of de Sitter solutions is obtained by transforming to synchronous coordinate systems. The representation of de Sitter solutions got by S. Hawking and G.Ellis [6] and E. Shrödinger [8] is a special case of the obtained one.

With applying the method of embedding into the 5-dimensional flat space, the properties of obtained representation of de Sitter models are studied. The expressions for embedding of these solutions are found. The outcomes are explained and the physical sense of integration function is ascertained. It is shown that the matching of de Sitter solution and the Tolman world is possible. The problem of time is considered. In Refs. [3, 6, 7] it was shown, that the de Sitter Universe can be described by three different times. In the present paper it is proved that in the de Sitter Universe there are infinitely many such times, thus it is no way to introduce the notion of universal cosmological time in the de Sitter world.

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