UDC 530.12

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INHOMOGENEOUS COSMOLOGICAL MODELS BASED ON THE STEPHANI SOLUTION

A class of spherically symmetric cosmological models based on the Stephani solution for the universe with shiftless perfect fluid is examined. Such models are close to the FLRW models and differ by a small deviation in spatial curvature while the dynamic Friedman equation is conserved. Two subclasses of models for positive and negative spatial curvature are built without concretization of the state equation in the center. The number of models with concrete state equations in the symmetry center are constructed and investigated for the positive spatial curvature. Cosmological parameters, such as Hubble parameter and deceleration parameter, are found and discussed for these models. Almost all of the models (except the ones with radiation and with the medium where gravity is higher than that of the dust, but lower than that of the radiation) have negative pressure at the early period of time, and all of these models tend to zero pressure during their evolution. The analysis of expressions for the deceleration parameter of these models showed that such parameter can be negative only for the models including phantom energy or quintessence, as in the FLRW models.

Key words: Einstein equations, universe accelerated expansion, Stephani soluiton, inhomogeneous cosmological models.

У роботі досліджено клас сферично симетричних космологічних моделей, базованих на розв’язку Стефані, який описує всесвіт, що заповнений беззсувною ідеальною рідиною. Розглянуто неоднорідні моделі, які є близькими до FLRW–моделей, але відрізняються від останніх просторовою кривиною. Побудовано підкласи розв’язків для позитивної та негативної просторової кривизни без конкретизації рівняння стану в центрі симетрії. Побудовано та досліджено ряд моделей з конкретизованим рівнянням стану в центрі симетрії. Для досліджуваних моделей отримано вирази для космологічних параметрів, таких як параметр Хаббла й параметр сповільнення, та розглянуто їх поведінку. Всі моделі (за винятком моделей із випромінюванням і моделей із параметром стану між пилом та випромінюванням) мають на початку еволюції від’ємний тиск, який згодом прямує до нуля. Аналіз виразів для параметра сповільнення даних моделей показав, що прискорене розширення всесвіту можливо тільки для моделей, які містять фантомну енергію або квінтесенцію, як і в моделях FLRW.

Ключові слова: рівняння Ейнштейна, прискорене розширення всесвіту, розв’язок Стефані, неоднорідні космологічні моделі

В работе исследован класс сферически симметричных космологических моделей основанных на решении Стефани, описывающем вселенную, заполненную бессдвиговой идеальной жидкостью. Рассмотрены неоднородные модели, близкие к моделям FLRW. Данные модели отличаются от FLRW–моделей пространственной кривизной. Для отрицательной и положительной пространственной кривизны построены подклассы моделей без конкретизации уравнения состояния в центре. Построен и исследован ряд моделей с конкретными уравнениями состояния в центре симметрии. Для полученных моделей исследовано поведение космологических параметров, таких как параметр Хаббла и параметр замедления. Все модели (за исключением моделей с излучением и моделей с параметром состояния между пылью и излучением) имеют на ранних стадиях эволюции отрицательное давление, которое с течением времени стремится к нулю. Анализ выражений для параметра замедления данных моделей показал, что ускоренное расширение вселенной возможно только для моделей, включающих фантомную энергию или квинтэссенцию, как и в FLRW–моделях.

Ключевые слова: уравнения Эйнштейна, ускоренное расширение вселенной, решение Стефани, неоднородные космологические модели.

1. Introduction

Discovery of the accelerated expansion of the universe [1] has led to a reconsideration of the standard model of the universe. But a new model ΛCDM raises new questions which are not covered by the standard model. In this connection, other models are actively being built at the present time. Models with homogeneous energy density and inhomogeneous pressure are especially interesting among them; such models are called Stephani cosmological ones. The assumption of homogeneity is a first approximation which was introduced to simplify Einstein equations. According to some authors, deviation from this assumption gives the possibility to explain accelerated expansion of the universe [2, 3]. It was noted that some models, based on the Stephani solution were in good agreement with the observational data.

The Stephani universe is an inhomogeneous solution for perfect fluid with zero shear, zero rotation and nonzero expansion. This solution was examined and discussed in numerous papers. More complete review may be found in [4-8]. Also, the Stephani solution has the FLRW solution as a limit and that is why it is very attractive in application to the cosmological models. Quite successfully, cosmological models based on the Stephani solution were built by Wesson & Ponce de Leon [9], Dabrowski [10], Sussman [11] Barrett & Clarkson [12], Stelmach & Jackaka [3]. In this paper we obtained and examined two types of exact Stephani solutions, which describe cosmological models and include models that were mentioned above.

2. The Stephani universe

The Stephani universe is a solution of the Einstein field equations with a perfect fluid source. The energy–momentum tensor is $T^{\alpha\beta} = (\varepsilon + p)u^\alpha u^\beta - pg^{\alpha\beta}$ where the energy density is $\varepsilon = \varepsilon(t)$, the pressure is $p = p(R,t)$, u^α is the four–velocity, $g^{\alpha\beta}$ is the metric tensor. In this paper we use the form of the Stephani solution that was obtained by mass–function method [13] in the paper [14] and that is more convenient for work, in our opinion. So, the metric of the spherically symmetric Stephani universe is:

$$dS^2 = \frac{\dot{r}^2}{r^2\psi^2}dt^2 - r^2\left(d\chi^2 + d\sigma^2\right), \tag{1}$$

$$r = 2(e^{\chi+\eta} - \zeta e^{-\chi-\eta})^{-1}, \tag{2}$$

$$\zeta = \psi^2 - \frac{1}{3}\varepsilon \tag{3}$$

where $\eta = \eta(t)$, $\psi = \psi(t)$, $\dot{r} = \partial r/\partial t$, $d\sigma^2 = d\theta^2 + \sin^2\theta d\phi^2$. The energy density and the pressure are connected by equation:

$$p = -\varepsilon - \frac{\dot{\varepsilon}}{3}\frac{r}{\dot{r}}. \tag{4}$$

χ may be chosen in such a way that the spatial part of the solution is conformal to one of three homogeneous and isotropic spaces:

$$dS^2 = \frac{\dot{r}^2(\chi,t)}{r^2(\chi,t)\psi^2(t)}dt^2 - r^2(\chi,t)\left(d\chi^2 + \begin{Bmatrix}\sin^2\chi\\ \sinh^2\chi\\ \chi^2\end{Bmatrix}d\sigma^2\right), \tag{5}$$

where for the closed universe:

$$r(\chi,t) = e^{-\eta(t)}\left[\cos^2\frac{\chi}{2} - \zeta(t)e^{-2\eta(t)}\sin^2\frac{\chi}{2}\right]^{-1}, \tag{6}$$

for the open universe:

$$r(\chi,t) = e^{-\eta(t)}\left[\cosh^2\frac{\chi}{2} - \zeta(t)e^{-2\eta(t)}\sinh^2\frac{\chi}{2}\right]^{-1}, \tag{7}$$

and for the flat universe:

$$r(\chi,t) = e^{-\eta(t)}\left[1 - \zeta(t)e^{-2\eta(t)}\frac{\chi^2}{4}\right]^{-1} \tag{8}$$

The Stephani universe contains three arbitrary functions depending on time – $\varepsilon(t)$, $\zeta(t)$, $\eta(t)$. $\zeta(t)$ is of the greatest interest. The analysis of invariants of the spatial curvature tensor of the metric (1) shows that the invariants depend on the arbitrary function $\zeta(t)$ only. The scalar curvature and the Kretschmann scalar are:

$$R = -6\zeta(t), \tag{9}$$

$$R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} = 12\zeta^2(t). \tag{10}$$

Thereby spatial curvature depends on $\zeta(t)$ only. The type of space (closed, open or flat) is determined by the sign of $\zeta(t)$. The fact that spatial curvature depends on an arbitrary function and thereby may change its sign with time was discussed before [4], [15]. It means that the spatial curvature is characterized by $\zeta(t)$ and $\zeta(t)$ completely determines the curvature. In this connection the flat universe is described by $\zeta = 0$. We choose $\chi = \ln 1/R$ to make the spatial part of the metric (1) flat:

$$dS^2 = \frac{1}{\dot{\eta}^2\psi^2}dt^2 - e^{-2\eta}(dR^2 + R^2 d\sigma^2). \tag{11}$$

In view of the arbitrariness of ψ and η, we can choose them as $1/\dot{\eta}$ and $\ln 1/a(t)$ ($a(t)$ is an arbitrary function), respectively, the metric (11) become:

$$dS^2 = dt^2 - a^2(dR^2 + R^2 d\sigma^2). \tag{12}$$

But this is the well known FLRW metric. Therefore it makes no sense to consider the flat Stephani universe, because it is the same that the FLRW one and, of course, some observation data are in a good agreement with it.

$\psi(t)$ makes sense of the critical energy density. When spatial curvature is zero, then $\zeta(t)=0$, according to (9). In accordance with (3):

$$\psi^2(t)=\frac{1}{3}\varepsilon_c(t). \tag{13}$$

3. FLRW as a special case of the Stephani universe

The FLRW solution is a special case of the Stephani universe. It is a well known fact [4, 11]. In this chapter we show the forms of arbitrary functions in our parameterizations, which transform the Stephani universe to the FLRW.

As it was mentioned above, the flat case of the Stephani universe is the FLRW one. It is easy to show that the Stephany universe is transformed to the FLRW (open and closed), when arbitrary functions have the following forms:

$$\begin{aligned} &\psi=\dot{a}/a, \quad \eta=\ln 1/a, \\ &\zeta>0: \quad \zeta=1/a^2, \qquad \chi=\ln\coth R/2, \\ &\zeta<0: \quad \zeta=-1/a^2, \qquad \chi=\ln\cot R/2, \end{aligned} \tag{14}$$

First of all, it is no sense in building a flat model based on the Stephani universe, because it is FLRW. If the universe is described by the Stephani solution, it will be curved.

It is interesting, in our view, to describe a curved FLRW model with some deviations, that lead to the Stephani universe. We take arbitrary functions in the form (14). We make an assumption that $a(t)$ is the FLRW scale factor. So, formally it is a FLRW solution. The energy density is taken in the form α/a^{n+2}, where α is a constant, n – some real number. So (3) becomes:

$$\frac{\dot{a}^2}{a^2}-\zeta=\frac{\alpha}{a^{n+2}}. \tag{15}$$

First, the case $\zeta=-1/a^2$ (closed model) is considered. The deviation from the FLRW model is achieved by adding a summand β/a^{k+2} (where β is a constant, k is a real number) to the energy density. We make the same deviation in ζ in order to save the Friedman equation (15):

$$\frac{\dot{a}^2}{a^2}+\frac{1}{a^2}+\frac{\beta}{a^{k+2}}=\frac{\alpha}{a^{n+2}}+\frac{\beta}{a^{k+2}}. \tag{16}$$

Thereby the energy density is:

$$\varepsilon=\frac{\alpha}{a^{n+2}}+\frac{\beta}{a^{k+2}}, \tag{17}$$

and spatial scalar curvature is:

$$\zeta=-\frac{1}{a^2}-\frac{\beta}{a^{k+2}}. \tag{18}$$

The expression for $r(\chi, t)$ may be obtained:

$$r = a\left[1 + \frac{\beta}{a^k}\sin^2\frac{\chi}{2}\right]^{-1}, \tag{19}$$

and the model metric is:

$$dS^2 = \left(1 + (k+1)W\right)^2 \left(1 + W\right)^{-2} dt^2 - a^2 (1+W)^{-2} (d\chi^2 + \sin^2\chi d\sigma^2), \tag{20}$$

where $W = \frac{\beta}{a^k}\sin^2\frac{\chi}{2}$.

For the case $\zeta = 1/a^2$ (an open model) equation (3) becomes:

$$\frac{\dot{a}^2}{a^2} - \frac{1}{a^2} - \frac{\beta}{a^{k+2}} = \frac{\alpha}{a^{n+2}} - \frac{\beta}{a^{k+2}}. \tag{21}$$

The energy density and spatial scalar curvature are:

$$\varepsilon = \frac{\alpha}{a^{n+2}} - \frac{\beta}{a^{k+2}}, \tag{22}$$

$$\zeta = \frac{1}{a^2} + \frac{\beta}{a^{k+2}}. \tag{23}$$

An expression for $r(\chi, t)$ becomes:

$$r = a\left[1 - \frac{\beta}{a^k}\sinh^2\frac{\chi}{2}\right]^{-1}, \tag{24}$$

and metric for this case is:

$$dS^2 = \left(1 - (k+1)V\right)^2 \left(1 - V\right)^{-2} dt^2 - a^2 (1-V)^{-2} (d\chi^2 + \sinh^2\chi d\sigma^2), \tag{25}$$

$V = \frac{\beta}{a^k}\sinh^2\frac{\chi}{2}$.

For the open universe the metric always has a singularity. Therefore, further we will work with the closed models. Constants β and α are arbitrary. It is worth noting, that these constants can be chosen small, and, in this case, the universe with small deviation from FLRW may be obtained.

As a result, the built class is closed to the FLRW. The Friedman equation is conserved. The barotropic equation of state in the symmetry center is determined by n and k. A very popular Dabrowski's model II [10], [3] is a partial case of our model when $k = -1$.

4. Pressure and energy density for $n = 1$–type models

$n = 1$–type models are two–component models where one of the components (n) is fixed and it is the cold matter and the other component (k) may be varied and determines the behavior of spatial curvature. The pressure and energy density for these models are described in this chapter. The expression for pressure can be obtained by substitution of (17) and (19) into the (4). It is

$$p = \left[3(1+(k+1)W)\right]^{-1}\left[\alpha((n-1)-3(k+1)W)a^{-(n+2)} + \beta((k-1)-3(k+1)W)a^{-(k+2)}\right] \qquad (26)$$

The behavior of $p(\chi, a(t))$ was studied for different k that correspond to different types of the second component. In our consideration: $k=-5$ for phantom energy; $k=-1$ for quintessence; $k=0$ for a medium where positive and negative gravity are absent; $k=1/4$ for a medium where gravity is lower than that of the dust; $k=1$ for dust; $k=7/4$ for a medium where gravity is higher than that of the dust, but lower than that of the radiation; $k=2$ for radiation.

It follows from the analysis that almost all of these models have negative pressure at early times, but soon enough pressure becomes zero. Exceptions are models for radiation and for medium where gravity is higher than that of the dust, but lower than that of the radiation. In these models a sign of the pressure depends on the spatial coordinate χ and scale factor $a(t)$. For some early period of time pressure is positive close to the symmetry centre and becomes negative at some χ. But during the evolution, pressure becomes zero, it tends to $+0$ unlike the other models, where it tends to -0. The model with phantom energy has singularity of pressure at early time. As regards the energy density, it decreases with time for all models (with the exception of phantom energy).

5. Observational parameters

The Hubble parameter H and deceleration parameter q are discussed in this chapter. Following Ellis [16] H and q are defined as

$$H \equiv \frac{1}{l}\frac{dl}{d\tau}, \quad q \equiv -l\left(\frac{dl}{d\tau}\right)^{-2}\frac{d^2 l}{d\tau^2}, \qquad (27)$$

where τ is a proper time measured along the particle world line, and l is some representative length along the particle world–line. We should transform the metric (1) to the following view if we want to use the definition (27):

$$ds^2 = d\tau^2 - l^2(\tau,\chi)(d\chi^2 + f^2(\chi)d\sigma^2), \qquad (28)$$

where $f(\chi) = \chi, \sin(\chi), \sinh(\chi)$ for $\zeta = 0$, $\zeta < 0$, and $\zeta > 0$ respectively; τ and l are defined as follows

$$d\tau = \frac{\dot{r}(\chi,t)}{r(\chi,t)\psi(t)}dt, \quad l(\tau,\chi) = \frac{r(t(\tau,\chi),\chi)}{f(\chi)}. \qquad (29)$$

In general, for the metric (1) H and q have the view

$$H = \psi(t), \quad q = -1 - \frac{\dot{H}}{H}\frac{r(t,\chi)}{\dot{r}(t,\chi)} \qquad (30)$$

For the closed model described by the metric (20) the Hubble parameter is

$$H = \frac{\dot{a}}{a}. \qquad (31)$$

The deceleration parameter is

$$q = -\frac{a(t)\ddot{a}(t)}{\dot{a}^2(t)} \cdot \frac{1+W}{1+(k+1)W} - \frac{k \cdot W}{1+(k+1)W}. \tag{32}$$

For the open model, that is described by the metric (25) the Hubble parameter is the same as (32) and the deceleration parameter is

$$q = -\frac{a(t)\ddot{a}(t)}{\dot{a}^2(t)} \cdot \frac{1-V}{1-(k+1)V} - \frac{k \cdot V}{1-(k+1)V}. \tag{33}$$

So, as for open models, so for closed models the Hubble parameter is the same as for FLRW (it is obviously from (31)). But q is not as simple as H; it is a complicated expression containing the dependence on the spatial coordinate χ.

In the previous chapter it was shown that pressure can be negative at some evolution period for these models. The question may appear: does it mean that deceleration parameter can be negative without phantom energy, quintessence or some another exotic matter, but due to the cold matter or radiation? Answer is: no. From (16) and (33) it follows that

$$q = -\frac{\beta}{a^k}\sin^2\frac{\chi}{2}\left(1+(k+1)\frac{\beta}{a^k}\sin^2\frac{\chi}{2}\right)^{-1}\left[k - \frac{n}{2}\left(1-\frac{a^n}{\alpha}\right)^{-1}\left(\frac{a^k}{\beta}\sin^{-2}\frac{\chi}{2} - 1\right)\right]. \tag{34}$$

There are two chances to make $q < 0$.

The first one is when $k > n\left[2\left(1 - a^n/\alpha\right)\right]^{-1}\left[(a^k/\beta)\sin^{-2}(\chi/2) - 1\right]$ and $1 + (k+1)\beta a^{-k}\sin^2(\chi/2) > 0.$

The second one is when $k < n\left[2\left(1 - a^n/\alpha\right)\right]^{-1}\left[(a^k/\beta)\sin^{-2}(\chi/2) - 1\right]$ and $1 + (k+1)\beta a^{-k}\sin^2(\chi/2) < 0.$

It can be seen, that the deceleration parameter can be negative only if k or n is negative, and that means the presence of some kind of phantom energy or quintessence. Thereby, this type of models can not explain the accelerated expansion of the Universe without some exotic types of matter.

6. Conclusions

Two models close to the FLRW and based on the Stephany solution are constructed. The models are built for the closed and open space. In these models the Friedman equation is conserved. A small deviation of FLRW energy density induces deviation in spatial curvature that, leads to effect of pressure inhomogeneity. Models for the open space contain singularity in their metrics, that is why they have not been described. Models for the closed space are considered for two–component models, where one of the component is the cold matter and another component may be varied (we consider phantom energy, quintessence, a medium where positive and negative gravity are absent, a medium where gravity is lower than that of the dust, dust, a medium where gravity is higher than that of the dust, but lower than that of the radiation, and radiation). Almost all of the models (except for models with radiation and with the medium where gravity is higher than that of the dust, but lower than that of the radiation) have negative pressure at the early period of time, and all of these models tend to zero pressure during their evolution. All of these models (except for phantom energy where singularity exists in pressure) have energy density that decreases with time inversely to the scale factor $a(t)$. The analysis of the expressions for the deceleration parameter for these models shows that this parameter can be negative only for the models, which include phantom energy or quintessence, as in the FLRW models.

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Received 19.04.2015.