ISSN 2408-9419. Вісник Дніпропетровського університету. Серія «Фізика. Радіоелектроніка». Вип. 23(2) Т.24, 2016
Visnik Dnìpropetrovs'kogo unìversitetu. Serìâ Fìzika, radìoelektronika. Issue 23(2) V. 24, 2016

D. O. Orlyans'ka, O. Yu. Orlyansky*, S. S. Protsenko

Oles Honchar Dnipropetrovsk National University, Dnipro, Ukraine
**e-mail: olegorl1@gmail.com*

THE SCHWARZSCHILD STATIC SOLUTIONS AND THE CRITERION OF COSMOLOGICAL EXPANSION

Physical consequences of matching the exterior and interior Schwarzschild solutions of General relativity with a cosmological constant are considered. It is shown that, if one interprets the obtained static model as some stationary distribution of matter, the estimate criterion of cosmological expansion can be supposed. According to this criterion, a spherically-symmetric configuration consisting of a usual and dark matter can not be gravitationally bound if its average density does not exceed double density of dark energy. If the average density of matter is less than twice the density of dark energy, a cosmological constant factor is dominant. In this case, a spherically symmetric cluster of matter (e.g., a cluster of galaxies) or uniformly expands, or lose external elements, depending on the degree of heterogeneity of the matter distribution. The proposed expansion criterion helps to understand the large-scale structure of the Universe better. Areas of the early Universe with lower density must expand with greater acceleration. As a result, the difference of densities will increase and lead to the formation of voids and surrounding denser structures.

Keywords: General relativity, cosmological constant, the Schwarzschild solutions, Universe accelerated expansion, criterion of cosmological expansion.

1. Introduction

In 1916, one hundred years ago, Karl Schwarzschild [1, 2] obtained the first exact solutions of the Einstein equations without a cosmological constant. Observation of more than decade specifies that the Universe is expanded with acceleration that can be explained with existence of a cosmological constant [3–5]. It is considered that the account of a cosmological constant is necessary in a cosmology, at the voids behaviour description, clusters of galaxies. At studying of stars, compact astrophysical objects cosmological constant consideration is not necessarily. However, in the paper we show that viewing the matching of the exterior and interior Schwarzschild solutions with a cosmological constant allows gaining estimate criterion of cosmological expansion, as effect of stability condition violation. The determining factor is not the size of the astrophysical object, but its density.

2. The exterior and interior Schwarzschild solutions with a cosmological constant

The Einstein equations with a cosmological constant Λ have the following view:

$$R^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu}R - \delta^{\mu}_{\nu}\Lambda = 8\pi\gamma T^{\mu}_{\nu}, \tag{1}$$

where the speed of light is chosen as unity, γ – gravitational constant, Λ – cosmological constant. We consider the spherically symmetric static metric in the standard form:

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)} - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{2}$$

Following Schwarzschild, for the interior solution we use an energy-momentum tensor for a perfect fluid with nonzero components $T^0_0 = \varepsilon$, $T^1_1 = T^2_2 = T^3_3 = -p$ where

$p(r)$ is the pressure and ε is the constant mass-energy density of matter. For star model construction, Schwarzschild has joined the exterior and interior solutions using conditions that the metric coefficients and their first derivatives are continuous on the joint surface. As it turned out, these conditions agree with the Lichnerowicz-Darmois conditions [6, 7] for line elements in form (2). The exterior or vacuum Schwarzschild solution $(\varepsilon = 0,\ p = 0)$ of the Einstein's equations (1) with a cosmological constant was found by Kottler [8] and has the form

$$ds_{out}^2 = \left(1 - \frac{r_g}{r} - \frac{\Lambda r^2}{3}\right)dt^2 - \frac{dr^2}{1 - \frac{r_g}{r} - \frac{\Lambda r^2}{3}} - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \qquad (3)$$

where $r_g = 2\gamma m$ – gravitational radius.

For the internal solution with constant density of energy ε and cosmological constant we have:

$$ds_{in}^2 = \left(A - B\sqrt{1 - \frac{r^2}{a^2}}\right)^2 dt^2 - \frac{dr^2}{1 - \frac{r^2}{a^2}} - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \qquad (4)$$

where A and B are integration constants, $a^2 = 3/\left(8\pi\gamma\varepsilon + \Lambda\right)$.
The pressure and energy density are:

$$8\pi\gamma p(r) = \frac{1}{a^2}\frac{3B\sqrt{1 - \frac{r^2}{a^2}} - A}{A - B\sqrt{1 - \frac{r^2}{a^2}}} + \Lambda,\ 8\pi\gamma\varepsilon = \frac{3}{a^2} - \Lambda. \qquad (5)$$

The constants A and B can be determined by considering the Lichnerowicz-Darmois boundary conditions. Let us suppose $r = R = const$ at the boundary. Then

$$a^2 = \frac{1}{\frac{r_g}{R^3} + \frac{\Lambda}{3}},\ B = \frac{\frac{r_g}{2R^3} - \frac{\Lambda}{3}}{\frac{r_g}{R^3} + \frac{\Lambda}{3}},\ A = \frac{\frac{3}{2}\frac{r_g}{R^3}\sqrt{1 - \frac{r_g}{R} - \frac{\Lambda R^2}{3}}}{\frac{r_g}{R^3} + \frac{\Lambda}{3}}. \qquad (6)$$

The first equality (6) is satisfied identically if we take into account $m = 4/3\pi R^3\rho$ where ρ is the density of matter.

3. Criterion of cosmological expansion

For building physically reasonable models, it is necessary to fulfill the stability conditions, in particular: $p'(r) < 0$, $p(r) > 0$.
These conditions have a simple physical sense. Pressure in sphere should be positive and decrease from the centre to a surface:

$$8\pi\gamma p'(r)=\frac{-2ABr}{a^4\left(A-B\sqrt{1-\frac{r^2}{a^2}}\right)^2\sqrt{1-\frac{r^2}{a^2}}}<0. \quad (7)$$

As follows from the third equality of Eq. (6), $A>0$. Hence, for the inequality (7) it is necessary that $B>0$. Then from the second equality (6) we find:

$$\frac{r_g}{2R^3}>\frac{\Lambda}{3}. \quad (8)$$

Inequality violation can be interpreted as the occurrence of a tendency to the expansion in the static solution, the expansion being related to a cosmological constant. As $r_g=\frac{2\gamma m}{c^2}=\frac{2\gamma}{c^2}\frac{4\pi}{3}R^3\rho$ where c is speed of light, from (8) we find:

$$\rho>\frac{\Lambda c^2}{4\pi\gamma}=2\rho_\Lambda. \quad (9)$$

In the last expression that the density ρ_Λ of dark energy is expressed through the cosmological constant as follows: $\rho_\Lambda=\Lambda c^2/8\pi\gamma$. Using the last data WMAP [9] for best fit WMAP + eCMB + BAO + H_0, the density of dark energy $\rho_\Lambda=6.44\cdot10^{-27}$ kg m^{-3}, and according to Planck data [10], we have $\rho_\Lambda=5.96\cdot10^{-27}$ kg m^{-3}.
There is a limiting density which is double density of dark energy. We will use the term evolutionary density $\rho_{ev}=2\rho_\Lambda$ for it. According to observational data, the evolutionary density ρ_{ev} is approximately in the range from $1.2\cdot10^{-26}$ kg m^{-3} to $1.3\cdot10^{-26}$ kg m^{-3}.
If the density is less than ρ_{ev}, the homogeneous static configuration is impossible. If the density of object is more than ρ_{ev}, it will be inconvertible and forces of gravitation will dominate over the factor of cosmological expansion.
It should be noted that the Friedmann equation with cosmological term for the second time derivative of the scale factor is

$$\frac{\ddot{a}}{a}=-\frac{4\pi\gamma}{3c^2}(\varepsilon+3p)+\frac{\Lambda c^2}{3}. \quad (10)$$

When the pressure of matter $p=0$, from (10) we have:

$$\frac{\ddot{a}}{a}=-\frac{4\pi\gamma}{3}(\rho-2\rho_\Lambda). \quad (11)$$

Sign of the acceleration of the expansion in the homogeneous cosmological models is determined only by the value of the density. The critical value again coincides with the evolutionary density $\rho_{ev}=2\rho_\Lambda$.

4. Conclusions

Let us notice that the same criterion was discussed in papers by Chernin et al. [11 – 13] where it has been obtained or from the classical physics limit for the case of weak fields or as interpretation of the Friedman equation with the cosmological term. The authors introduced the concept of a Hubble cell and zero gravity radius where the force of gravity and the cosmological repulsion are equal. In paper by Byrd et al. [14] the dynamical structure of a gravitating system within dark energy based on the Newtonian approach was explained with using this terminology. In general, the papers of last years relating to the study of groups of galaxies, and in particular, the Local Group [14, 15], based on observational data still use Newton's approach to the cosmological expansion. In our paper we obtain the same result, but based on the exact solutions of the equations of general relativity. On the one hand, in the investigation of the Universe expansion it is better to use a relativistic theory than the theory of Newton. On the other hand, this work shows that the first exact static Schwarzschild solutions obtained a hundred years ago can be useful in cosmology even today. The proposed criterion of evolutionary expansion helps to better understanding of the large-scale structure of the Universe. Areas of the early Universe with low density must expand with great acceleration. As a result, the difference of densities will increase and lead to the formation of voids and the surrounding denser structures.

Acknowledgments

We thank Prof. Korkina M.P. for useful discussions.

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Received 5.12.2016