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ON THE QUANTIZATION OF BLACK HOLES

To build a quantum model of a black hole, we introduce a modified description of classical space-time black hole. The Lagrangian formalism of vacuum gravitational field in spherically symmetric space-time consisting of two R- and T- regions is developed. A coordinate system, in which the components of the metric depend on the time coordinate in the T-region and on the spatial coordinate in the R-region, is introduced. Then we construct the Hamiltonian and mass functions corresponding to the evolution coordinates (*t* and *r*) in each of the regions. Their Poisson brackets are proportional to the Hamiltonian constraint. Next, we construct the quantum Hamiltonian and the mass operator. Their commutators are proportional to the Hamilton operator. The system of the DeWitt equation and the equation for mass operator eigenvalues, together with the compatibility condition, allow us to find the wave functions in each region. These parts of wave function form the common wave function of the black hole with the continuous spectrum of masses *m*. As an additional condition limiting the arbitrariness in the choice of solutions, we use the regularity condition for the wave function on the black hole horizon.

Keywords: black holes, mass function, Hamiltonian constraint, quantization, mass operator and Hamilton operator, compatibility condition.

1. Introduction

The quantization of a black hole (BH), based on the idea of quantizing the horizon area of BH was put forward by Bekenstein in [1]. It turns out that the horizon area of a nonextremal BH behaves in a sense as an adiabatic invariant [2]. This approach has been developed in many papers [3-6], and leads to the discrete mass spectrum. However, the geometrodinamical approaches [7, 8] to the quantum theory of BH gives a continuous spectrum (see also [9]). To incorporate these approaches different ways have been proposed [10, 11].

In this paper, based on a simple geometrodinamical approach, using DeWitt equation and quantum mass operator is constructed the quantum model BH with continuous mass spectrum.

2. Classical description of BH

The action for gravitational field in the space-time V^4 has the form

$$S = -\frac{c^3}{16\pi\kappa}\int_{V^4}\sqrt{-g}Rd^4x\,. \qquad (1)$$

For the spherically symmetric space-time with the metric

$$ds^2 = h\left(x^0,r\right)\left(dx^0\right)^2 - g\left(x^0,r\right)dr^2 - R^2\left(x^0,r\right)d\sigma^2 \qquad (2)$$

($d\sigma^2 = d\theta^2 + \sin^2\theta d\alpha^2$), the action, after reduction, can be written as

$$S = \frac{c^3}{2\kappa}\int_{V^2}\sqrt{hg}\left\{\frac{R}{g}R_{,1}\left(\ln(hR)\right)_{,1} - \frac{R}{h}R_{,0}\left(\ln(gR)\right)_{,0} + 1\right\}dx^0 dr. \tag{3}$$

Here $R_{,0} = \partial A / \partial x^0$, $R_{,1} = \partial A / \partial r$. Information about the space structure is contained in the formula: $(\nabla R)^2 = g^{ab} R_{,a} R_{,b}$. The surface $R(x^0, r) = R_g = const$, for which $(\nabla R)^2 = 0$, divides V^4 on T- and R-regions. Here with in T – region $(\nabla R)^2 > 0$, and in R-region $(\nabla R)^2 < 0$. We choose the coordinates in which the metric is dependent of the coordinate r in the R-region, and the coordinate x^0 in the T-region:

$$\begin{aligned} ds_-^2 &= h_-\left(x^0\right)\left(dx^0\right)^2 - g_-\left(x^0\right)dr^2 - R_-^2\left(x^0\right)d\sigma^2, \\ ds_+^2 &= h_+\left(r\right)\left(dx^0\right)^2 - g_+\left(r\right)dr^2 - R_+^2\left(r\right)d\sigma^2. \end{aligned} \tag{4}$$

Then, the initial action can be written as $S = S_- + S_+ = \int L_- dx^0 + \int L_+ dr$, where $L_\pm$ – effective Lagrangian in the R- and T - regions:

$$L_- = \frac{lc^3}{2\kappa}\sqrt{h_- g_-}\left\{1 - \frac{R_-}{h_-}R_{-\,,0}\left(\ln\left(g_- R_-\right)\right)_{,0}\right\}, \quad l = r_2 - r_1 \tag{5}$$

$$L_+ = \frac{lc^3}{2\kappa}\sqrt{h_+ g_+}\left\{1 + \frac{R_+}{g_+}R_{+,1}\left(\ln\left(h_+ R_+\right)\right)_{,1}\right\}, \quad l = x_2^0 - x_1^0 \tag{6}$$

Let us consider T-region. It is convenient to introduce new variables

$$h_- = \frac{n+u}{n-u}N^2, \quad g_- = \frac{n-u}{n+u}, \quad R_- = n+u. \tag{7}$$

Then, the Lagrangian (5) takes the form

$$L_- = \frac{lc^3}{2\kappa}\left\{\frac{1}{N_-}\left(\dot{u}^2 - \dot{n}^2\right) + N_-\right\} \tag{8}$$

The relation $P_N = \partial L_- / \partial \dot{N} = 0$ provides the primary constraint. Next, we construct the momentums

$$P_u = \frac{\partial L_-}{\partial \dot{u}} = \frac{lc^3}{\kappa N}\dot{u} = \text{const}, \quad P_n = \frac{\partial L_-}{\partial \dot{n}} = -\frac{lc^3}{\kappa N}\dot{n} = \text{const} \tag{9}$$

From the Lagrangian (8) we find one of the components of the Einstein equations

$$\frac{\partial L_-}{\partial N_-} = \frac{lc^3}{2\kappa}\left\{-\frac{1}{N_-^2}\left(\dot{u}^2 - \dot{n}^2\right) + 1\right\} = 0 \text{ - it is a secondary constraint.} \tag{10}$$

Therefore, the Hamiltonian vanishes in the weak sense (in Dirac's sense):

$$H = \frac{\kappa N_-}{2lc^3}\left(P_u^2 - P_n^2 - \frac{l^2 c^6}{\kappa^2}\right) \approx 0 \tag{11}$$

We introduce the mass function $M = \left(c^2/2\kappa\right) R\left(1+\gamma^{ab} R_{,a} R_{,b}\right)$. Here it takes the form

$$M = \frac{c^2}{2\kappa}\left(n+u+\frac{1}{N^2}(n-u)(\dot{n}+\dot{u})^2\right) = \frac{c^2}{2\kappa}\left(n+u+\frac{\kappa^2}{l^2c^6}(n-u)\left(P_u+P_n\right)^2\right) \qquad (12)$$

It can be shown that following relation holds $\partial M/\partial x^0 = \{M,H\} = -\left(\kappa/l^2c^4\right)H \approx 0$, where $\{M,H\}$ is the Poisson bracket. Thus, M is a motion integral. Excluding N from (8) and (10), we find $L_- = \left(lc^3/\kappa\right)\sqrt{\dot{u}^2-\dot{n}^2}$. Hence, it follows action for gravitational field in minisuperspace

$$S_- = \int L_- dx^0 = \frac{lc^3}{\kappa}\int\sqrt{du^2-dn^2} = \int d\Omega \qquad (13)$$

where $d\Omega^2 = du^2 - dn^2$ is the metric minisuperspace. The geodesic equations for this metric are equivalent to the remaining Einstein equations.

3. Quantum description of BH

We define the momentum operators in the standard way

$$P_u = -i\hbar\frac{\partial}{\partial u}, \quad P_n = -i\hbar\frac{\partial}{\partial n}. \qquad (14)$$

Taking into account (11) and (12), the operators of Hamilton and the mass, have the form

$$\hat{H} = -\frac{N_-\kappa\hbar^2}{2lc^3}\left(\frac{\partial^2}{\partial u^2}-\frac{\partial^2}{\partial n^2}+\frac{l^2}{l_{pl}^4}\right), \hat{M} = \frac{c^2}{2\kappa}\left(n+u-\frac{\kappa^2\hbar^2}{l^2c^6}(n-u)\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial n}\right)^2\right) \qquad (15)$$

where $l_{pl}^2 = \hbar\kappa/c^3$.

Now, we can write the Hamiltonian, the DeWitt equation, the mass operator and the equation $\left(\hat{M}-m\right)\Psi = 0$ in the light coordinates $\xi = n - u$. $\eta = n + u$:

$$\hat{H} = -\frac{N_-\hbar}{2}\left(\frac{4l_{pl}}{\lambda}\frac{\partial^2}{\partial\xi\partial\eta}+\frac{\lambda}{l_{pl}}\right), \qquad \frac{\partial^2\Psi}{\partial\xi\partial\eta}-\frac{\lambda^2}{4l_{pl}^2}\Psi = 0,$$
$$\hat{M} = \frac{c^2}{2\kappa}\left(\eta-\frac{4l_{pl}^2}{\lambda^2}\xi\frac{\partial^2}{\partial\xi^2}\right), \qquad \xi\frac{\partial^2}{\partial\xi^2}\Psi = \frac{\lambda^2}{4l_{pl}^2}\left(\eta-2l_{pl}\mu\right)\Psi. \qquad (16)$$

It turns out that $\left[\hat{H},\hat{M}\right] \approx H \approx 0$. To find the common wave function of equations $\hat{H}\Psi = 0$ and $\left(\hat{M}-m\right)\Psi = 0$, need to use their compatibility condition. We obtain

$$\Psi = xz^{-1/2}\left\{C_1J_1\left(\lambda\sqrt{z}\right)+C_2Y_1\left(\lambda\sqrt{z}\right)\right\}, \quad x = \xi/l_{pl} = gR/l_{pl},$$
$$z = \left(\xi/l_{pl}^2\right)\left(2l_{pl}\mu-\eta\right) = \left(u/l_{pl}-\mu\right)^2-\left(n/l_{pl}-\mu\right)^2 = \left(gc^3R^2/\hbar\kappa\right)\left(2\kappa m/c^2R-1\right), \qquad (17)$$

Where $J_1(z)$ and $Y_1(z)$ are the Bessel functions of the first and second kind. For the regularity Ψ on the horizon $R = 2\kappa m/c^2$, we require $C_2 = 0$. As a result, the wave function of BH with the mass *m* in T-region takes the form

$$\Psi = C_1 g^{1/2}\left(\frac{2\kappa m}{c^2 R} - 1\right)^{-1/2} J_1\left(\sqrt{g\left(\frac{2\kappa m}{c^2 R} - 1\right)}\right). \tag{18}$$

For the R-region, taking into account L_+ in (7), and using *r* as evolution coordinate, we come to the same result. Thus, the common for the space-time V^4, the wave function has the form (18) and corresponds to the BH state with the mass *m*.

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