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ON OPTIMAL CONTROL PROBLEM IN COEFFICIENTS FOR NONLINEAR ELLIPTIC VARIATIONAL INEQUALITIES

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In this paper we study an optimal control problem for a nonlinear elliptic variational inequality with generalized solenoidal coefficients which we adopt as controls in $L^\infty(\Omega)$. We prove the existence of optimal solution of the stated problem.

Key words: monotone equations, elliptic variational inequalities, control in coefficients, Sobolev spaces.

1. Introduction

The aim of this paper is to prove an existence result for optimal control problem in coefficients of a nonlinear elliptic variational inequality using the direct method of calculus of variation and the compensated compactness lemma. As François Murat showed in 1970 (see [15], [16]), the optimal control problems in coefficients have no solution in general even for linear elliptic equations. It turns out that this feature is typical for the majority of optimal control problems in coefficients. Besides, this fact is not just a mathematical problem, but it is also very restrictive in view of numerical applications.

Let Ω be a fixed non-empty open subset of $\mathbb{R}^N$ with a smooth boundary. The optimal control problem we consider in this paper is to minimize the discrepancy between a given distribution $z_\partial \in L^p(\Omega)$ and the solution of a nonlinear elliptic variational inequality by choosing an appropriate matrix of coefficients

$$\mathcal{U} \in L^\infty(\Omega; \mathbb{R}^{N\times N}).$$

Namely, we consider the following minimization problem:

$$L(\mathcal{U}, y) = \int_\Omega |y(x) - z_\partial(x)|^p \, dx \to \inf \tag{1.1}$$

subject to

$$\mathcal{U} \in M_p^{\alpha,\beta}(\Omega),\ y \in K, \tag{1.2}$$

$$\langle -\mathrm{div}\left(\mathcal{U}(x)\,[(\nabla y)^{p-2}]\nabla y\right) + |y|^{p-2}y, v - y\rangle_V \geq \langle f, v - y\rangle_V \ \forall\, v \in K, \tag{1.3}$$

where K is a closed convex subset of $V = W_0^{1,p}(\Omega)$, $f \in L^q(\Omega)$ is a fixed function, and the matrix $[\eta^{p-2}]$ is defined as follows

$$[\eta^{p-2}] = \operatorname{diag}\{|\eta_1|^{p-2}, |\eta_2|^{p-2}, \dots, |\eta_N|^{p-2}\} \quad \forall \eta \in \mathbb{R}^N. \tag{1.4}$$

We seek a matrix of coefficients $\mathcal{U} \in M_p^{\alpha,\beta}(\Omega)$ such that the corresponding weak solution $y_{\mathcal{U},f}$ of (1.1)–(1.3) would be as close to the desired state z_∂ as possible.

Note that since the range of optimal control problems in coefficients is very wide, including as well the optimal shape design problems, optimization of certain evolution systems, some problems originating in mechanics and others, this topic has been widely studied by many authors. We mainly could mention Allaire [2], Buttazzo & Dal Maso [3], [4], Calvo-Jurado & Casado-Diaz [5], [6], [7], Lions [12], Linvinov [13], Lurie [14], Murat [16], Murat & Tartar [17], Raytum [18], Serovaĭskiĭ [19], Tiba [20], Mel'nik & Zgurovsky [21]. However, to the best knowledge of author, the existence of the optimal solutions in coefficients to variational inequalities has not been considered in literature.

As was mentioned above, the principal feature of such problems is the fact that there does not exist an optimal solution in general (see, e.g., [3], [5], [16], [18]). So here we have a typical situation for the general optimal control theory. Namely, the original control object is described by well-posed boundary value problem, but the associated optimal control problem is ill-posed and requires relaxation.

Taking this fact into account, we restrict the problem (1.1)–(1.3) by introducing the so-called solenoidal controls $\mathcal{U} \in U_{sol}$ (for comparison, see [9], [10]). Notice that this class of admissible controls does not belong to the Sobolev space $W^{1,\infty}(\Omega)$, but still is a uniformly bounded subset of $L^\infty(\Omega)$. We give the precise definition of such controls in Section 3 and prove that in this case the original optimal control problem admits at least one solution. Note that we do not involve the homogenization method and the relaxation procedure in this process.

2. Notation and Preliminaries

In this section we introduce some notation and preliminaries that will be useful later on.

For two real numbers $1 < p < +\infty$, $1 < q < +\infty$ such that $1/p + 1/q = 1$, the space $W_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the classical Sobolev space $W^{1,p}(\Omega)$, while $W^{-1,q}(\Omega)$ is the dual space of $W_0^{1,p}(\Omega)$.

For any vector field $\vec{v} \in \mathbf{L}^q(\Omega) = [L^q(\Omega)]^N$, the divergence is an element of the space $W^{-1,q}(\Omega)$ defined by the formula

$$\langle \operatorname{div} \vec{v}, \varphi \rangle_{W_0^{1,p}(\Omega)} = -\int_\Omega (\vec{v}, \nabla\varphi)_{\mathbb{R}^N}\, dx, \quad \forall \varphi \in W_0^{1,p}(\Omega), \tag{2.1}$$

where $\langle \cdot, \cdot \rangle_{W_0^{1,p}(\Omega)}$ denotes the duality pairing between $W^{-1,q}(\Omega)$ and $W_0^{1,p}(\Omega)$, and $(\cdot, \cdot)_{\mathbb{R}^N}$ denotes the scalar product of two vectors in $\mathbb{R}^N$.

A vector field $\vec{v}$ is said to be solenoidal, if $\operatorname{div}\vec{v} = 0$. For any vector field $\vec{v} \in \mathbf{L}^q(\Omega)$ the relations

$$\langle \operatorname{curl}\vec{v}, \varphi\rangle^{i\,j}_{W_0^{1,\,p}(\Omega)} = -\int_\Omega \left(v_i \frac{\partial \varphi}{\partial x_j} - v_j \frac{\partial \varphi}{\partial x_i} \right) dx, \; \forall\, \varphi \in W_0^{1,\,p}(\Omega), \quad i,j = 1,\dots,N,$$

define a skew-symmetric matrix $\operatorname{curl}\vec{v}$, with elements in $W^{-1,\,q}(\Omega)$. A vector field $\vec{v}$ is said to be vortex-free, if $\operatorname{curl}\vec{v} = 0$. We say that a vector field $\vec{v} \in \mathbf{L}^p(\Omega)$ is potential, if $\vec{v}$ can be represented in the form $\vec{v} = \nabla u$, where $u \in W^{1,\,p}(\Omega)$. Obviously, any potential vector is vortex-free.

Monotone operators. Let α and β be constants such that $0 < \alpha \le \beta < +\infty$. We define the class $M_p^{\alpha,\beta}(\Omega)$ as a set of all symmetric matrices $\mathcal{U}(x) = \{a_{i\,j}(x)\}_{1\le i,j\le N}$ in $L^\infty(\Omega; \mathbb{R}^{N\times N})$ such that the following conditions of growth, monotonicity, and strong coercivity are fulfilled:

$$|a_{i\,j}(x)| \le \beta \quad \text{a.e. in } \; \Omega, \; \forall\, i,j \in \{1,\dots,N\}, \tag{2.2}$$

$$\big(\mathcal{U}(x)([\zeta^{p-2}]\zeta - [\eta^{p-2}]\eta), \zeta - \eta\big)_{\mathbb{R}^N} \ge 0 \quad \text{a.e. in } \; \Omega, \; \forall\, \zeta, \eta \in \mathbb{R}^N, \tag{2.3}$$

$$\big(\mathcal{U}(x)[\zeta^{p-2}]\zeta, \zeta\big)_{\mathbb{R}^N} = \sum_{i,j=1}^N a_{ij}(x)|\zeta_j|^{p-2}\,\zeta_j\,\zeta_i \ge \alpha\,|\zeta|_p^p \quad \text{a.e. in } \; \Omega, \tag{2.4}$$

where $|\eta|_p = \left(\sum\limits_{k=1}^N |\eta_k|^p \right)^{1/p}$ is a Hölder norm of order p in $\mathbb{R}^N$ and the matrix $[\zeta^{p-2}]$ is defined in (1.4).

Remark 2.1. It is easy to see that $M_p^{\alpha,\beta}(\Omega)$ is a nonempty subset of the space $L^\infty(\Omega; \mathbb{R}^{N\times N})$ and its typical representatives are diagonal matrices of the form

$$\mathcal{U}(x) = \operatorname{diag}\{\delta_1(x), \delta_2(x), \dots, \delta_N(x)\},$$

where $\alpha \le \delta_i(x) \le \beta$ a.e. in $\Omega\; \forall\, i \in \{1,\dots,N\}$. Indeed, in this case the conditions (2.2) and (2.4) obviously hold. To verify the monotonicity property (2.3), let us fix two arbitrary vectors ζ and η in $\mathbb{R}^N$. Then

$$\begin{aligned}
\big(\mathcal{U}(x)([\zeta^{p-2}]\zeta - [\eta^{p-2}]\eta), \zeta - \eta\big)_{\mathbb{R}^N} &= \big([\zeta^{p-2}]\mathcal{U}(x)\zeta, \zeta\big)_{\mathbb{R}^N} \\
&- \big([\zeta^{p-2}]\mathcal{U}(x)\zeta, \eta\big)_{\mathbb{R}^N} - \big([\eta^{p-2}]\mathcal{U}(x)\eta, \zeta\big)_{\mathbb{R}^N} + \big([\eta^{p-2}]\mathcal{U}(x)\eta, \eta\big)_{\mathbb{R}^N} \\
&= \sum_{i=1}^N \delta_i(x)|\zeta_i|^{p-2}\zeta_i^2 - \sum_{i=1}^N \delta_i(x)|\zeta_i|^{p-2}\zeta_i\eta_i \\
&- \sum_{i=1}^N \delta_i(x)|\eta_i|^{p-2}\zeta_i\eta_i + \sum_{i=1}^N \delta_i(x)|\eta_i|^{p-2}\eta_i^2 \\
&= \sum_{i=1}^N \delta_i(x)|\zeta_i|^{p-2}\zeta_i(\zeta_i - \eta_i) - \sum_{i=1}^N \delta_i(x)|\eta_i|^{p-2}\eta_i(\zeta_i - \eta_i) \\
&= \sum_{i=1}^N \delta_i(x)\left(|\zeta_i|^{p-2}\zeta_i - |\eta_i|^{p-2}\eta_i\right)(\zeta_i - \eta_i).
\end{aligned} \tag{2.5}$$

As a result, the inequality (2.3) is a direct consequence of the well-known estimates

$$(|a|^{p-2}a - |b|^{p-2}b)(a-b) \geq 2^{2-p}|a-b|^p,\ p \geq 2, \quad \forall\, a,\ b \in \mathbb{R}, \tag{2.6}$$

$$(|a|^{p-2}a - |b|^{p-2}b)(a-b) \geq (|a|+|b|)^{p-2}\,|a-b|^2,\ 1 < p \leq 2, \quad \forall\, a,\ b \in \mathbb{R},. \tag{2.7}$$

Lemma 2.1. *For every fixed control $\mathcal{U} \in M_p^{\alpha,\beta}(\Omega)$ an operator $A_{\mathcal{U}} : V \to V^*$ defined as*

$$\langle A_{\mathcal{U}}(y), v\rangle_V = \sum_{i,j=1}^N \int_\Omega \left(a_{ij}(x) \left| \frac{\partial y}{\partial x_j} \right|^{p-2} \frac{\partial y}{\partial x_j} \right) \frac{\partial v}{\partial x_i}\, dx + \int_\Omega |y|^{p-2} y\, v\, dx,$$

is strictly monotone, coercive and semicontinuous (here by the semicontinuity property we mean that the scalar function $t \to \langle A_{\mathcal{U}}(y+tv), w\rangle_V$ is continuous for all $y,\ v,\ w \in V$).

Proof. To begin with, we prove the coercivity property of the operator $A_{\mathcal{U}}$, i.e. we prove that $\frac{\langle A_{\mathcal{U}}(y), y\rangle_V}{\|y\|_V} \to +\infty$, as $\|y\|_V \to \infty$. Let $\mathcal{U} \in M_p^{\alpha,\beta}(\Omega)$ be a fixed matrix. Then

$$\langle A_{\mathcal{U}}(y), y\rangle_V = \sum_{i,j=1}^N \int_\Omega \left(a_{ij}(x) \left| \frac{\partial y}{\partial x_j} \right|^{p-2} \frac{\partial y}{\partial x_j} \right) \frac{\partial y}{\partial x_i}\, dx + \int_\Omega |y|^p\, dx = I_1 + I_2.$$

Due to (2.4) we have $I_1 \geq \alpha \displaystyle\int_\Omega |\nabla\, y|_p^p\, dx$. Therefore,

$$\begin{aligned} I_1 + I_2 &\geq \min\{\alpha, 1\} \int_\Omega \left(|y|^p + |\nabla\, y|_p^p \right)\, dx \\ &= \min\{\alpha, 1\} \|y\|_V^p = \gamma(\|y\|_V)\, \|y\|_V, \end{aligned} \tag{2.8}$$

where $\gamma(s) = \min\{\alpha, 1\} s^{p-1} \to \infty$ as $s \to \infty$. Hence, the operator $A_{\mathcal{U}}$ is coercive.

In order to prove the monotonicity of $A_{\mathcal{U}}$, we make use of the estimate (2.3) and the strict monotonicity of the term $f(y) = |y|^{p-2}y$ with respect to estimations (2.6) and (2.7). As a result, we have

$$\begin{aligned} \langle A_{\mathcal{U}}(y) - A_{\mathcal{U}}(v), y - v\rangle_V &= \int_\Omega \left(|y|^{p-2}y - |v|^{p-2}v \right)(y-v)\, dx \\ &+ \int_\Omega \sum_{i,j=1}^N a_{ij}(x) \left(\left| \frac{\partial y}{\partial x_j} \right|^{p-2} \frac{\partial y}{\partial x_j} - \left| \frac{\partial v}{\partial x_j} \right|^{p-2} \frac{\partial v}{\partial x_j} \right) \left(\frac{\partial y}{\partial x_i} - \frac{\partial v}{\partial x_i} \right) dx \\ &= \int_\Omega \left(\mathcal{U}(x)([(\nabla y)^{p-2}]\nabla y - [(\nabla v)^{p-2}]\nabla v), \nabla y - \nabla v \right)_{\mathbb{R}^N}\, dx \\ &+ \int_\Omega \left(|y|^{p-2}y - |v|^{p-2}v \right)(y-v)\, dx \geq \int_\Omega \left(|y|^{p-2}y - |v|^{p-2}v \right)(y-v)\, dx > 0, \\ &\forall\, y \neq v \text{ a.e. in } \Omega. \end{aligned} \tag{2.9}$$

The semicontinuity property of $A_{\mathcal{U}}$ is a direct consequence of the continuity of the following functions

$$z_1(t) := \int_\Omega \sum_{i,j=1}^{N} \big(\mathcal{U}[\nabla(y+tv)^{p-2}]\nabla(y+tv), \nabla w\big)_{\mathbb{R}^N}\, dx = \int_\Omega \Phi_1(x,t)\, dx,$$
$$z_2(t) := \int_\Omega |y+tv|^{p-2}(y+tv)w\, dx = \int_\Omega \Phi_2(x,t)\, dx,$$

Since $|\Phi_1(x,t)| \le \Phi_1^0(x)$ and $|\Phi_2(x,t)| \le \Phi_2^0(x)$, by Hölder inequality it follows that $\Phi_1^0(x) \in L^1(\Omega)$ and $\Phi_2^0(x) \in L^1(\Omega)$.

As $t \to 0$, we have

$$\Phi_1(x,t) \to \Phi_1(x,0) = \sum_{i,j=1}^{N} \big(\mathcal{U}[\nabla(y)^{p-2}]\nabla y, \nabla w\big) \text{ for a.e. } x,$$
$$\Phi_2(x,t) \to \Phi_2(x,0) = |y|^{p-2}yw \text{ for a.e. } x,$$
$$\int_\Omega \Phi_1(x,0)\, dx + \int_\Omega \Phi_2(x,0)\, dx = \langle A_{\mathcal{U}}(y), w\rangle_V = z_1(0) + z_2(0).$$

Hence it is sufficient to cite Lebesgue's dominated theorem to obtain the required relations $\lim\limits_{t\to 0} z_1(t) = z_1(0)$ and $\lim\limits_{t\to 0} z_2(t) = z_2(0)$. The proof is complete. □

Elliptic variable inequalities. Following Lions [11], let us cite some well known results concerning solvability and uniqueness and smoothness properties for nonlinear variational inequalities which we use in the sequel.

Theorem 2.1. *[11, Theorem 8.2] Let V be a Banach space and $K \subset V$ be a closed convex subset. Suppose also that $A : K \to V^*$ is a nonlinear operator and $f \in V^*$ is a given element of the dual space. The following variational problem: to find an element $y \in K$ such that*

$$\langle A(y), v - y\rangle_V \ge \langle f, v - y\rangle_V, \quad \forall v \in K, \tag{2.10}$$

admits at least one solution provided the following conditions:

1. *operator A is pseudomonotone, i.e. it is bounded and if $y_k \to y$ weakly in V, $y_k, y \in K$ and $\limsup_{k\to\infty}\langle A(y_k), y_k - y\rangle_V \le 0$, then*
$$\liminf_{k\to\infty}\langle A(y_k), y_k - v\rangle_V \ge \langle A(y), y - v\rangle_V,\ \forall v \in V.$$
2. *operator A is coercive, i.e. there exists an element $v_0 \in K$ such that*
$$\frac{\langle A(y), y - v_0\rangle_V}{\|y\|_V} \to +\infty \text{ as } \|y\|_V \to \infty,\ y \in K$$

Theorem 2.2. *[11, Theorem 8.3] If the operator $A : K \to V^*$ in Theorem 2.1 is strictly monotone on K, then variational inequality* (2.10) *admits a unique solution.*

The pseudomonotonicity property plays the key role in solvability of the problem (2.10). The following result concerns with the sufficient conditions for fulfillment of this property.

Proposition 2.1. [11, Proposition 2.5] For a nonlinear operator $A : V \to V^*$ the following implication takes place: A is a bounded monotone semicontinuous operator $\Rightarrow$ A is a pseudomonotone operator.

Referring to Lions [11], we make use the following assumptions.

Hypothesis 1. There exists a reflexive Banach space X such that $X \subset V^*$, the imbedding $X \hookrightarrow V^*$ is continuous, and X is dense in V^*.

Hypothesis 2. There can be found a duality mapping $J : X \to X^*$ such that $\forall\, y \in K, \forall\, \varepsilon > 0$ there exists an $y_\varepsilon \in K$ such that $A(y_\varepsilon) \in X$ and $y_\varepsilon + \varepsilon J(A(y_\varepsilon)) = y$.

Theorem 2.3. *[11, Theorem 8.7] Assume that the Hypotheses 1 and 2 hold true*[1]*. Let operator $A : V \to V^*$ be monotone, semicontinuous, bounded and satisfy assumption 2 of Theorem 2.1. Then the inclusion $f \in X$ implies that any solution y of variational inequality* (2.10) *is such that $A(y) \in X$.*

3. Setting of the optimal control problem

The optimal control problem we consider in this paper is to minimize the discrepancy between a given distribution $z_\partial \in L^p(\Omega)$ and the solution $y = y_{\mathcal{U},f}$ of the variational inequality (1.2)–(1.3) by choosing an appropriate matrix of coefficients $\mathcal{U} \in L^\infty(\Omega; \mathbb{R}^{N\times N})$. Namely, we consider the minimization problem in the form (1.1)–(1.3).

Let ξ_1, ξ_2 be given functions of $L^\infty(\Omega)$ such that $0 < \xi_1(x) \le \xi_2(x)$ a. e. in Ω. Let $\{Q_1, \dots, Q_N\}$ be a collection of nonempty compact subsets of $W^{-1,q}(\Omega)$.

To define the class of admissible controls, we introduce two sets

$$U_b = \Big\{ \mathcal{U} = [a_{i\,j}] \in M_p^{\alpha,\beta}(\Omega) \Big| \, \xi_1(x) \le a_{i\,j}(x) \le \xi_2(x) \\ \text{a.e. } x \in \Omega, \ \forall\, i, j = 1, \dots, N \Big\}, \tag{3.1}$$

$$U_{sol} = \Big\{ \mathcal{U} = [\vec{a}_1, \dots, \vec{a}_N] \in M_p^{\alpha,\beta}(\Omega) \Big| \operatorname{div} \vec{a}_i \in Q_i, \ \forall\, i = 1, \dots, N \Big\}, \tag{3.2}$$

assuming that the intersection $U_b \cap U_{sol} \subset L^\infty(\Omega; \mathbb{R}^{N\times N})$ is a nonempty set.

Definition 3.1. We say that a matrix $\mathcal{U} = [a_{i\,j}]$ is an admissible control to the variational inequality (1.2)–(1.3) if $\mathcal{U} \in U_{ad} := U_b \cap U_{sol}$.

[1] (see also [11, Theorem 8.8.])

Remark 3.1. We suppose that the set of admissible controls U_{ad} is sufficiently rich, otherwise, the optimal control problem

$$L(\mathcal{U}, y) = \int_\Omega |y(x) - z_\partial(x)|^p \, dx \to \inf, \tag{3.3}$$

$$\mathcal{U} \in U_{ad}, \ y \in K, \tag{3.4}$$

$$\langle -\mathrm{div}\, \big(\mathcal{U}(x) \, [(\nabla y)^{p-2}] \nabla y\big) + |y|^{p-2} y, v - y\rangle_V \geq \langle f, v - y\rangle_V \ \forall\, v \in K, \tag{3.5}$$

becomes trivial. Notice also that this class of admissible controls does not belong to $W^{1,\infty}(\Omega)$ or to the Sobolev space $W^{1,q}(\Omega)$, but still is a uniformly bounded subset of $L^\infty(\Omega)$.

The existence of admissible controls is important both from a theoretical and an applicational point of view. Usually controls of this type arise in the optimization of materials (represented by the matrix $\mathcal{U}$). So this question is largely an open one, except for some special cases, and an affirmative answer is usually just put as a hypothesis (see [2], [6], [14]).

Taking this fact into account we can indicate the following set of admissible pairs to the optimal control problem (3.3)–(3.5):

$$\Xi = \{(\mathcal{U}, y) \in U_{ad} \times V \ \mid y \in K, (\mathcal{U}, y) \text{ are related by } (1.3)\,\}. \tag{3.6}$$

As an obvious consequence of Theorems 2.1, 2.2, Proposition 2.1 and Lemma 2.1, we have the following conclusion.

Proposition 3.1. For every control $\mathcal{U} \in M_p^{\alpha,\beta}(\Omega)$ and every $f \in L^q(\Omega)$ there exists a unique solution to the variational inequality (3.4)–(3.5).

Proof. Since the first assumption of Theorem 2.1 is obviously true, it remains to verify the condition 2 of that Theorem. Let us fix an arbitrary element $v_0 \in K$ and a matrix $\mathcal{U} \in M_p^{\alpha,\beta}(\Omega)$. For all $y \in K$ we have:

$$\begin{aligned}\langle A(\mathcal{U}, y), y - v_0\rangle_V &= \langle A(\mathcal{U}, y), y\rangle_V - \langle A(\mathcal{U}, y), v_0\rangle_V \\ &\geq \gamma(\|y\|_V)\, \|y\|_V - \langle A(\mathcal{U}, y), v_0\rangle_V \geq \gamma(\|y\|_V)\, \|y\|_V - |\langle A(\mathcal{U}, y), v_0\rangle_V|.\end{aligned} \tag{3.7}$$

Using the estimate (2.8), we obtain

$$\begin{aligned}|\langle A(\mathcal{U}, y), v_0\rangle_V| &= \left| \int_\Omega \big(\mathcal{U}[(\nabla y)^{p-2}]\nabla y, \nabla v_0\big)_{\mathbb{R}^N} \, dx + \int_\Omega |y|^{p-2} y v_0 \, dx \right| \\ &\{\text{ in view of } (2.2)\} \leq \beta \left| \int_\Omega \big([(\nabla y)^{p-2}]\nabla y, \nabla v_0\big)_{\mathbb{R}^N} \, dx \right| \\ &+ \left| \int_\Omega |y|^{p-2} y v_0 \, dx \right| \leq \beta \, \|\nabla v_0\|_{\mathbf{L}^p(\Omega)} \left(\int_\Omega \left|[(\nabla y)^{p-2}]\nabla y\right|_q^q \, dx \right)^{1/q} \\ &+ \|v_0\|_{L^p(\Omega)} \left(\int_\Omega \big(|y|^{p-1}\big)^q \, dx \right)^{1/q} = \{\text{ since } q = p/(p-1)\} \\ &= \beta \, \|v_0\|_V \left(\int_\Omega |\nabla y|_p^p \, dx \right)^{1/q} + \|v_0\|_{L^p(\Omega)} \|y\|_{L^p(\Omega)}^{p-1} \\ &\leq \max\{\beta, 1\} \, \|v_0\|_V \, \|y\|_V^{p-1}.\end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8), we come to the required conclusion

$$\frac{\langle A(\mathcal{U},y),y-v_0\rangle_V}{\|y\|_V} \geq \gamma(\|y\|_V) - \max\{\beta,1\}\,\|v_0\|_V\,\|y\|_V^{p-2}$$
$$= \|y\|_V^{p-1}\left(\min\{\alpha,1\} - \frac{\max\{\beta,1\}\|v_0\|_V}{\|y\|_V}\right) \to +\infty, \text{ as } \|y\|_V \to \infty. \quad (3.9)$$

□

As was mentioned in proposition 3.1, the set Ξ is nonempty. So, we adopt the following concept:

Definition 3.2. We say that a pair $(\mathcal{U}^0,y^0) \in L^\infty(\Omega;\mathbb{R}^{N\times N})\times V$ is an optimal solution to the problem (3.4)–(3.5) if $(\mathcal{U}^0,y^0)\in\Xi$ and $L(\mathcal{U}^0,y^0) = \inf\limits_{(\mathcal{U},y)\in\Xi} L(\mathcal{U},y)$.

The main question to be answered on the problem (3.3)–(3.5) is about solvability: does an optimal pair $(\mathcal{U}^0,y^0)$ in $L^\infty(\Omega;\mathbb{R}^{N\times N})\times W_0^{1,p}(\Omega)$ satisfying (3.3)–(3.5) exist? To begin with, we need the following result (see [9]):

Proposition 3.2. The set U_{ad} is sequentially compact with respect to the weak-$*$ topology of $L^\infty(\Omega;\mathbb{R}^{N\times N})$.

Proof. Let $\{\mathcal{U}_k = [\vec{a}_{1\,k},\dots,\vec{a}_{N\,k}]\}_{k\in\mathbb{N}} \subset U_{ad}$ be an arbitrary sequence of admissible controls. Since $U_{ad}\subset U_b$ and U_b is the sequentially weakly-$*$ compact subset of $L^\infty(\Omega;\mathbb{R}^{N\times N})$, we may suppose that there exist a matrix $\mathcal{U}_0 = [\vec{a}_{1\,0},\dots,\vec{a}_{N\,0}]\in U_b$ and elements $f_i\in Q_i$ $i=1,\dots,N$ such that

$$\int_\Omega (\vec{a}_{i\,k},\varphi)_{\mathbb{R}^N}\,dx \to \int_\Omega (\vec{a}_{i\,0},\varphi)_{\mathbb{R}^N}\,dx, \text{ as } k\to\infty,$$
$$\forall\varphi\in\mathbf{L}^1(\Omega) = [L^1(\Omega)]^N,\ \forall\, i=1,2,\dots,N, \quad (3.10)$$

$$\text{and } \operatorname{div}\vec{a}_{i\,k}\to f_i \text{ strongly in } W^{-1,q}(\Omega), \text{ as } k\to\infty\ \forall\, i=1,\dots,N. \quad (3.11)$$

It remains to prove that $\operatorname{div}\vec{a}_{i\,0} = f_i$ for all $i=1,\dots,N$. To do this, we choose φ in (3.10) as a potential vector, that is, $\varphi = \nabla v$, where $v\in W_0^{1,p}(\Omega)$. Then, the relation (3.11) implies $\int_\Omega (\vec{a}_{i\,k},\nabla v)_{\mathbb{R}^N}\,dx = -\langle \operatorname{div}\vec{a}_{i\,k}, v\rangle_{W_0^{1,p}(\Omega)} \to -\langle f_i,v\rangle_{W_0^{1,p}(\Omega)}$, as $k\to\infty$, $\forall i=1,\dots,N$. Using this and relation (3.10), we finally get

$$\lim_{k\to\infty}\int_\Omega (\vec{a}_{i\,k},\nabla v)_{\mathbb{R}^N}\,dx = \int_\Omega (\vec{a}_{i\,0},\nabla v)_{\mathbb{R}^N}\,dx$$
$$= -\langle \operatorname{div}\vec{a}_{i\,0},v\rangle_{W_0^{1,p}(\Omega)} = -\langle f_i,v\rangle_{W_0^{1,p}(\Omega)} \quad \forall\, i=1,\dots,N.$$

As a result, we have $\mathcal{U}_0 = [\vec{a}_{1\,0},\dots,\vec{a}_{N\,0}]\in U_{sol}$. This concludes the proof. □

4. Existence of optimal solutions

In order to discuss the existence of solutions for the problem (3.3)–(3.5), we make use of the following result (for comparison see [22]).

Lemma 4.1. *[9] Let $\{\vec{f}_k\}_{k\in\mathbb{N}} \subset \mathbf{L}^q(\Omega)$, $\{\vec{g}_k\}_{k\in\mathbb{N}} \subset \mathbf{L}^p(\Omega)$ be the bounded sequences of vector-functions such that $\vec{f}_k \rightharpoonup \vec{f}_0$ in $\mathbf{L}^q(\Omega)$ and $\vec{g}_k \rightharpoonup \vec{g}_0$ in $\mathbf{L}^p(\Omega)$. If $\left\{\operatorname{div}\vec{f}_k\right\}_{k\in\mathbb{N}}$ is compact with respect to the strong topology of $W^{-1,q}(\Omega)$, and $\operatorname{curl}\vec{g}_k = 0\ \forall\, k \in \mathbb{N}$, then*

$$\lim_{k\to\infty}\int_\Omega \phi\,\left(\vec{f}_k,\vec{g}_k\right)_{\mathbb{R}^N}\,dx = \int_\Omega \phi\,\left(\vec{f}_0,\vec{g}_0\right)_{\mathbb{R}^N}\,dx, \quad \forall\,\phi\in C_0^\infty(\Omega). \tag{4.1}$$

Now we are in a position to study the topological properties of the set $\Xi \subset L^\infty(\Omega;\mathbb{R}^{N\times N}) \times W_0^{1,p}(\Omega)$ of all admissible pairs to the optimal control problem (3.3)–(3.5). Let τ be the topology on the set $L^\infty(\Omega;\mathbb{R}^{N\times N}) \times W_0^{1,p}(\Omega)$ which we define as the product of the weak-$*$ topology of $L^\infty(\Omega;\mathbb{R}^{N\times N})$ and the weak topology of $W_0^{1,p}(\Omega)$.

Theorem 4.1. *Assume that for the set K in problem (3.4)–(3.4) the hypothesis 2 holds true provided $X = L^q(\Omega)$. Then for every $f \in L^q(\Omega)$ the set Ξ is sequentially τ-closed.*

Proof. Let $\{(\mathcal{U}_k, y_k)\}_{k\in\mathbb{N}} \subset \Xi$ be any τ-convergent sequence of admissible pairs to the problem (3.3)–(3.5). Let $(\mathcal{U}_0, y_0)$ be its τ-limit. Our aim is to prove that $(\mathcal{U}_0, y_0) \in \Xi$. Let us set

$$\begin{gathered} A(\mathcal{U},y) = -\operatorname{div}\,\big(\mathcal{U}(x)\,[(\nabla y)^{p-2}]\,\nabla y\big) + |y|^{p-2}\,y = A_1(\mathcal{U},y) + A_2(\mathcal{U},y),\\ A_1(\mathcal{U},y) = -\operatorname{div}\,\big(\mathcal{U}(x)\,[(\nabla y)^{p-2}]\,\nabla y\big) = -\operatorname{div} a(\mathcal{U}(x),\nabla y). \end{gathered}$$

By Proposition 3.2 and the initial assumptions, we have $\mathcal{U}_0 \in U_{ad}$. Therefore,

$$\mathcal{U}_k \rightharpoonup \mathcal{U}_0 = [\vec{a}_{1\,0},\dots,\vec{a}_{N\,0}] \quad \text{weakly-}*\quad \text{in}\quad L^\infty(\Omega, R^{N\times N}), \tag{4.2}$$

$$\operatorname{div}\vec{a}_{i\,k} \to \operatorname{div}\vec{a}_{i\,0} \text{ strongly in }\ W^{-1,q}(\Omega),\ \forall\, i = 1,\dots,N, \tag{4.3}$$

$$y_k \rightharpoonup y_0 \ \text{ in } W_0^{1,p}(\Omega). \tag{4.4}$$

Hence

$$\begin{gathered} \left\{[(\nabla y_k)^{p-2}]\,\nabla y_k\right\}_{k\in\mathbb{N}} \ \text{ is bounded in }\ \mathbf{L}^q(\Omega),\ q = p/(p-1),\\ \left\{|y_k|^{p-2}y_k\right\}_{k\in\mathbb{N}} \ \text{ is bounded in }\ L^q(\Omega), \end{gathered} \tag{4.5}$$

$$y_k \to y_0 \ \text{ strongly in }\ L^p(\Omega), \quad y_k(x) \to y_0(x) \text{ a.e. in } \Omega. \tag{4.6}$$

Then, by (4.6) and monotonicity of the function $g(\zeta) = |\zeta|^{p-2}\zeta$, we have $|y_k|^{p-2}y_k \to |y_0|^{p-2}y_0$ almost everywhere in Ω. Using this and (4.5), we conclude (see [11]): $|y_k|^{p-2}y_k \rightharpoonup |y_0|^{p-2}y_0$ in $L^q(\Omega)$. Since $f \in L^q(\Omega)$, in view of theorem 2.3 we have $-\operatorname{div}\,\big(\mathcal{U}_k\,[(\nabla y_k)^{p-2}]\,\nabla y_k\big) + |y_k|^{p-2}\,y_k \in L^q(\Omega)$, $\forall\, k \in \mathbb{N}$ and, hence, $-\operatorname{div} a(\mathcal{U}_k,\nabla y_k) \in L^q(\Omega)\ \forall\, k \in \mathbb{N}$. The sequence $\{y_k\}_{k\in\mathbb{N}}$ is bounded in the space $W_0^{1,p}(\Omega)$ due to the coercivity of the operator $A(\mathcal{U},y)$ (see (3.9)). Therefore, the sequence $\{\mathcal{U}_k[(\nabla y)^{p-2}]\nabla y\}_{k\in\mathbb{N}}$ is bounded in $\mathbf{L}^q(\Omega)$. So, passing to a subsequence, we may assume that there exists a vector-function $\vec{\xi} \in \mathbf{L}^q(\Omega)$ such that

$$a(\mathcal{U}_k,\nabla y_k) = \mathcal{U}_k[(\nabla y)^{p-2}]\nabla y =: \vec{\xi}_k \rightharpoonup \vec{\xi} \quad \text{in}\quad \mathbf{L}^q(\Omega). \tag{4.7}$$

In view of this and the fact that

$$\langle -\operatorname{div}\vec{\xi_k}, \varphi\rangle_V = \int_\Omega (\xi_k, \nabla\varphi)_{\mathbb{R}^N}\, dx \\ \to \int_\Omega (\xi, \nabla\varphi)_{\mathbb{R}^N}\, dx = \langle -\operatorname{div}\vec{\xi}, \varphi\rangle_V,\ \forall\, \varphi \in C_0^\infty(\Omega),$$

we have: $\operatorname{div}\vec{\xi_k} \to \operatorname{div}\vec{\xi}$ weakly in $W^{-1,q}(\Omega)$. The fact that $\operatorname{div}\vec{\xi_k} \in L^q(\Omega)$ $\forall\, k \in \mathbb{N}$ implies (see [1]): $\|\operatorname{div}\vec{\xi_k}\|_{L^q(\Omega)} = \|\operatorname{div}\vec{\xi_k}\|_{W^{-1,q}(\Omega)}$, $\forall\, k \in \mathbb{N}$. Hence, the sequence $\{\operatorname{div}\vec{\xi_k}\}_{k\in\mathbb{N}}$ is bounded in $L^q(\Omega)$. Therefore, due to the compactness of the embedding $L^q(\Omega) \hookrightarrow W^{-1,q}(\Omega)$, we may suppose that the strong convergence in $W^{-1,q}(\Omega)$ of this sequence takes place. In what follows, we show that $\vec{\xi} = \mathcal{U}_0[(\nabla y_0)^{p-2}]\nabla y_0$. To do so, we consider the scalar function

$$v(x) = (z, x)_{\mathbb{R}^N}, \tag{4.8}$$

where z is a fixed element of $\mathbb{R}^N$. Since the operator A_1 is monotone, it follows that for every $z \in \mathbb{R}^N$ and every positive function $\varphi \in C_0^\infty(\Omega)$, we have $\int_\Omega \varphi(x)\,(a(\mathcal{U}_k, \nabla y_k) - a(\mathcal{U}_k, \nabla v), \nabla y_k - \nabla v)_{\mathbb{R}^N}\ dx \geq 0$, or, taking into account (4.8), this inequality can be rewritten as

$$\int_\Omega \varphi(x)\,(a(\mathcal{U}_k, \nabla y_k) - a(\mathcal{U}_k, z), \nabla y_k - z)_{\mathbb{R}^N}\ dx \geq 0. \tag{4.9}$$

Our next intention is to pass to the limit in (4.9) as $k \to \infty$ using Lemma 4.1. Since

$$\left.\begin{aligned} -\operatorname{div} a(\mathcal{U}_k, \nabla y_k) \to -\operatorname{div}\vec{\xi}\ \text{strongly in } W^{-1,\,q}(\Omega),\\ \operatorname{curv}(\nabla\, y_k - z) = \operatorname{curv}\nabla\, y_k = 0, \quad \forall\, k \in \mathbb{N}, \end{aligned}\right\} \tag{4.10}$$

it remains to show that the sequence $\{\operatorname{div} a(\mathcal{U}_k, z)\}_{k\in\mathbb{N}}$ is compact with respect to the strong topology of $W^{-1,\,q}(\Omega)$.

Indeed, for every $\varphi \in C_0^\infty(\Omega)$, we have

$$\begin{aligned} &\langle -\operatorname{div} a(\mathcal{U}_k, z), \varphi\rangle_V = \int_\Omega (a(\mathcal{U}_k, z), \nabla\varphi)_{\mathbb{R}^N}\ dx \\ &= \int_\Omega \left(\mathcal{U}_k[z^{p-2}]z, \nabla\varphi\right)\ dx = \int_\Omega \left(\begin{bmatrix} (\vec{a}_{1\,k}(x), [z^{p-2}]z)_{\mathbb{R}^N} \\ \dots \\ (\vec{a}_{N\,k}(x), [z^{p-2}]z)_{\mathbb{R}^N} \end{bmatrix}, \nabla\varphi\right)_{\mathbb{R}^N} dx = \\ &= \int_\Omega \sum_{i=1}^N \left(\vec{a}_{i\,k}(x), [z^{p-2}]z\right)_{\mathbb{R}^N} \frac{\partial\varphi}{\partial x_i}\, dx = \int_\Omega \sum_{i=1}^N \sum_{j=1}^N a^k_{i\,j}(x) \frac{\partial\varphi}{\partial x_i} |z_j|^{p-2} z_j\, dx = \\ &= \sum_{j=1}^N |z_j|^{p-2} z_j \int_\Omega (\vec{a}_{j\,k}(x), \nabla\varphi)_{\mathbb{R}^N}\, dx = \sum_{j=1}^N |z_j|^{p-2} z_j\, \langle -\operatorname{div}\vec{a}_{j\,k}, \varphi\rangle_V = J_k. \end{aligned} \tag{4.11}$$

Then using (4.3), we get

$$\lim_{k\to\infty} J_k \;=\; \sum_{j=1}^{n} |z_j|^{p-2} z_j \lim_{k\to\infty} \langle -\mathrm{div}\, \vec{a}_{j\,k}, \varphi\rangle_V \;=\; \sum_{j=1}^{n} |z_j|^{p-2} z_j \,\langle -\mathrm{div}\, \vec{a}_{j\,0}, \varphi\rangle_V \,. \tag{4.12}$$

Making the converse transformations with (4.12) as we did it in (4.11), we come to the relation

$$\lim_{k\to\infty} \langle -\mathrm{div}\, a(\mathcal{U}_k, z), \varphi\rangle_V = \langle -\mathrm{div}\, a(\mathcal{U}_0, z), \varphi\rangle_V \,. \tag{4.13}$$

Since for every $i = 1, \dots, N$ the sequences $\{\mathrm{div}\, \vec{a}_{i\,k}\}_{k\in\mathbb{N}}$ are strongly convergent in $W^{-1,q}(\Omega)$, from (4.11)–(4.13) it follows that

$$\lim_{k\to\infty} \langle -\mathrm{div}\, a(\mathcal{U}_k, z), \varphi_k\rangle_V = \langle -\mathrm{div}\, a(\mathcal{U}_0, z), \varphi\rangle_V \tag{4.14}$$

for each sequence $\{\varphi_k\}_{k\in\mathbb{N}} \subset C_0^\infty(\Omega)$ such that $\varphi_k \rightharpoonup \varphi$ in $W_0^{1,p}(\Omega)$. Thus, summing up the above results, we obtain

$$\left.\begin{aligned} \mathrm{div}\, a(\mathcal{U}_k, z) &\to \mathrm{div}\, a(\mathcal{U}_0, z) \quad \text{strongly in } \; W^{-1,q}(\Omega), \\ a(\mathcal{U}_k, z) = \mathcal{U}_k[z^{p-2}]z &\rightharpoonup \mathcal{U}_0[z^{p-2}]z \quad\;\; \text{weakly-} * \;\text{ in } \; \mathbf{L}^\infty(\Omega). \end{aligned}\right\} \tag{4.15}$$

As a result, combining properties (4.10) and (4.15), it has been shown that all suppositions of Lemma 4.1 are fulfilled. So, taking into account (4.4), (4.10), (4.15), and passing to the limit in inequality (4.9) as $k \to \infty$, we get

$$\int_\Omega \varphi(x)\, (\xi - a(\mathcal{U}_0, z), \nabla\, y_0 - z)_{\mathbb{R}^N}\; dx \geq 0, \quad \forall z \in \mathbb{R}^N$$

for all positive $\varphi \in C_0^\infty(\Omega)$. After localization, we have $(\xi - a(\mathcal{U}_0, z), \nabla y_0 - z)_{\mathbb{R}^N} \geq 0$, for a.a. $x \in \Omega$, $\forall z \in \mathbb{R}^N$.

Remark 4.1. The operator $a(\mathcal{U}, \cdot) : \mathbb{R}^N \to \mathbb{R}^N$ is monotone and continuous. Indeed, for any sequence $\{u_k = [u_k^1, \dots, u_k^N]\} \subset \mathbb{R}^N$ such that $u_k \to u_0 = [u_0^1, \dots, u_0^N]$ in $\mathbb{R}^N$, it follows that $u_k^i \to u_0^i$ in $\mathbb{R}$, $\forall\, i = 1, \dots, N$. Then, it is easy to see that $|u_k^i|^{p-2} u_k^i \to |u_k^i|^{p-2} u_k^i$ in $\mathbb{R}$ $\forall\, i = 1, \dots, N$ and, therefore, $[(u_k)^{p-2}]u_k \to [(u_0)^{p-2}]u_0$ in $\mathbb{R}^N$. Then the monotonicity property immediately follows from the estimate (2.3).

Further, since the operator $a(\mathcal{U}, \cdot) : \mathbb{R}^N \to \mathbb{R}^N$ is monotone, and continuous, then in view of [8, Lemma III.1.3] it follows that

$$\xi = a(\mathcal{U}_0, \nabla y_0) = \mathcal{U}_0[(\nabla y_0)^{p-2}]\nabla y_0, \text{ for a.a. } x \in \Omega. \tag{4.16}$$

Now, we can pass to the limit in the variational inequality

$$\langle A(\mathcal{U}_k, y_k), y_k - v\rangle_V \leq \langle f, y_k - v\rangle_V, \; \forall\, v \in K, \tag{4.17}$$

using again for its left hand-side the Compensated Compactness Lemma 4.1.

$$\lim_{k\to\infty}\langle A(\mathcal{U}_k,y_k),y_k-v\rangle_V$$
$$=\lim_{k\to\infty}\langle \operatorname{div} a(\mathcal{U}_k,\nabla y_k),y_k-v\rangle_V+\lim_{k\to\infty}\langle |y_k|^{p-2}y_k,y_k-v\rangle_V$$
$$=\langle \operatorname{div} a(\mathcal{U}_0,\nabla y_0),y_0-v\rangle_V+\lim_{k\to\infty}\langle |y_k|^{p-2}y_k,y_k-v\rangle_V.$$

Since $|y_k|^{p-2}y_k \rightharpoonup |y_0|^{p-2}y_0$ in $\mathbf{L}^q(\Omega)$, in view of compactness of the embedding $L^q(\Omega)\hookrightarrow W^{-1,q}(\Omega)$ we have: $|y_k|^{p-2}y_k \to |y_0|^{p-2}y_0$ strongly in $W^{-1,q}(\Omega)$. Therefore, $\lim_{k\to\infty}\langle |y_k|^{p-2}y_k,y_k-v\rangle_V = \langle |y_0|^{p-2}y_0,y_0-v\rangle_V$. Passing to the limit in (4.17), we obtain $\langle A(\mathcal{U}_0,y_0),y_0-v\rangle_V \le \langle f,y_0-v\rangle_V$, $\forall v\in K$, Hence, the τ-limit pair (A_0,y_0) is admissible to the problem (3.3)–(3.5), and this concludes the proof. □

Now we can turn to the existence of optimal pairs.

Theorem 4.2. *Under the control admissibility hypothesis* ($U_{ad}=U_b\cap U_{sol}\neq\emptyset$), *the optimal control problem* (3.3)–(3.5) *admits at least one solution* $(\mathcal{U}^{opt},y^{opt})\in \Xi\subset L^\infty(\Omega;\mathbb{R}^{N\times N})\times W_0^{1,p}(\Omega)$ *for every* $f\in L^q(\Omega)$.

Proof. The control admissibility condition ensures the existence of a minimizing sequence $\{(\mathcal{U}_k,y_k)\in\Xi\}_{k\in\mathbb{N}}$, i.e. $\lim_{k\to\infty}L(\mathcal{U}_k,y_k)=\inf_{(\mathcal{U},y)\in\Xi}L(\mathcal{U},y)<+\infty$. Since the sequence of admissible controls $\{\mathcal{U}_k\in U_{ad}\}_{k\in\mathbb{N}}$ is bounded in the space $L^\infty(\Omega;\mathbb{R}^{N\times N})$ and using the arguments of Theorem 4.1 one can easily to show that the minimizing sequence is bounded in $L^\infty(\Omega;\mathbb{R}^{N\times N})\times W_0^{1,p}(\Omega)$ and hence, within a subsequence, there exists a pair $(\mathcal{U}^*,y^*)$ such that $\mathcal{U}_k\rightharpoonup\mathcal{U}^*$ weakly-$*$ in $L^\infty(\Omega,\mathbb{R}^{N\times N})$, $y_k\rightharpoonup y^*$ in $W_0^{1,p}(\Omega)$. By Theorem 4.1 the pair $(\mathcal{U}^*,y^*)$ is admissible to the problem (3.3)–(3.5). Moreover, since the cost functional L is lower semicontinuous, we get $L(\mathcal{U}^*,y^*)\le \liminf_{k\to\infty}L(\mathcal{U}_k,y_k)=\inf_{(\mathcal{U},y)\in\Xi}L(\mathcal{U},y)$. Hence, $(\mathcal{U}^*,y^*)$ is an optimal pair. □

Remark 4.2. The argumentas used in the proof of Theorem 4.2 is related to the so-called "direct method" of the Calculus of Variations which, roughly speaking, intends to construct a minimizing sequence $\{(\mathcal{U}_k,y_k)\in\Xi\}_{k\in\mathbb{N}}$.

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