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ON HENIG REGULARIZATION OF STATE-CONSTRAINED OPTIMAL CONTROL PROBLEM FOR THE P-LAPLACE EQUATION

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We study a Dirichlet optimal control problem for a quasilinear monotone p-Laplace equation with control and state constraints. The coefficient of the p-Laplacian, the weight u, we take as a control in $L^1(\Omega)$. We discuss a relaxation of such problem following the so-called Henig regularization scheme.

Key words: p-Laplace problem, optimal control, control in coefficients, relaxation, Henig dilating cone, existence result.

1. Introduction

The aim of this article is to analyze an optimal control problem for a nonlinear PDE with mixed boundary conditions where the coefficient of p-Laplacian operator we take as a control. As for the class of admissible controls, we consider it as a nonempty subset of $L^1(\Omega)$ with an empty topological interior. Such choice is motivated by needs of having good properties of solutions to the corresponding boundary value problem. Since an important matter for applications is to obtain a solution to a given boundary problem with desired properties, it leads to the reasonable questions: can we define an appropriate coefficient of p-Laplacian to minimize the discrepancy between a given displacement y_d and an expected solution to such problem.

The characteristic feature of OCP we deal with in this article, is the fact that the solutions of nonlinear boundary value problem should be restricted by some pointwise constraints in L^p-spaces. In fact, the ordering cone of positive elements in L^p-spaces is typically nonsolid, i.e. it has an empty topological interior. Following Lagrange multiplier rule, which gives a necessary optimality condition for local solutions to state constrained OCPs, the constraint qualifications such as the Slater condition or the Robinson condition should be applied in this case. However, these conditions cannot be verified for cones such as $L^p_+(\Omega)$ due to $\operatorname{int}\left(L^p_+(\Omega)\right) = \emptyset$. Therefore, our main intention in this article is to propose a suitable relaxation of the pointwise state constraints in the form of some inequality conditions involving a so-called Henig approximation $\left(L^p_+(\Omega)\right)_\varepsilon(B)$ of the ordering cone of positive elements $L^p_+(\Omega)$. Here, B is a fixed closed base of $L^p_+(\Omega)$. Due to fact that $L^p_+(\Omega) \subset \left(L^p_+(\Omega)\right)_\varepsilon(B)$ for all $\varepsilon > 0$, we can replace the cone $L^p_+(\Omega)$ by its approximation $\left(L^p_+(\Omega)\right)_\varepsilon(B)$. As a result, it leads to some relaxation of the inequality constraints of the considered problem, and, hence, to an approximation of the set of admissible pairs to OCP. The main issue is to show that admissibility and solvability of a given class of OCPs can be characterized by solving the corresponding Henig relaxed problems in the limit $\varepsilon \to 0$.

2. Definitions and Basic Properties

Let Ω be a bounded open connected subset of $\mathbb{R}^N$ $(N \geq 2)$. We assume that the boundary $\partial\Omega$ is Lipschitzian so that the unit outward normal $\nu = \nu(x)$ is well-defined for a.e. $x \in \partial\Omega$, where a.e. means here with respect to the $(N-1)$-dimensional Hausdorff measure. We also assume that the boundary $\partial\Omega$ consists of two disjoint parts $\partial\Omega = \Gamma_D \cup \Gamma_S$, where the sets Γ_D and Γ_N have positive $(N-1)$-dimensional measures, and Γ_N is now C^1.

Let p be a real number such that $2 \leq p < \infty$. By $W^{1,p}(\Omega)$ we denote the Sobolev space as the subspace of $L^p(\Omega)$ of functions y having generalized derivative Dy in $L^p(\Omega)$. It is well known that $W^{1,p}(\Omega)$ is a Banach space with respect to the norm (see [1, Theorem 4.14])

$$\|y\|_{W^{1,p}(\Omega)} = \left(\|y\|^p_{L^p(\Omega)} + \|Dy\|^p_{L^p(\Omega)}\right)^{1/p} = \left(\int_\Omega \left(|y|^p + |Dy|^p\right)\,dx\right)^{1/p}.$$

For any $y \in C^1(\overline{\Omega})$ we define the traces

$$\gamma_0(y) = y\,|_{\partial\Omega}\,, \quad \text{and} \quad \gamma_1(y) = \left.\frac{\partial y}{\partial\nu}\right|_{\partial\Omega}.$$

By [10, Theorem 8.3], these linear operators can be extended continuously to the whole of space $W^{1,p}(\Omega)$. We set $W^{1/q,p}(\partial\Omega) := \gamma_0\left[W^{1,p}(\Omega)\right]$ as closed subspace of $L^p(\partial\Omega)$, where $q = p/(p-1)$ is the conjugate of p. Moreover, the injection

$$W^{1/q,p}(\partial\Omega) \hookrightarrow L^p(\partial\Omega) \tag{2.1}$$

is compact.

Let $C_0^\infty(\mathbb{R}^N;\Gamma_D) = \left\{\varphi \in C_0^\infty(\mathbb{R}^N)\ :\ \varphi = 0 \ \text{ on } \Gamma_D\right\}$. We define the Banach space $W_0^{1,p}(\Omega;\Gamma_D)$ as the closure of $C_0^\infty(\mathbb{R}^N;\Gamma_D)$ with respect to the norm $\|y\|_{W^{1,p}(\Omega)}$. Let $W^{-1,q}(\Omega;\Gamma_D)$ be the dual space to $W_0^{1,p}(\Omega;\Gamma_D)$.

Throughout this paper, we use the notation $\mathbb{W}_p(\Omega) := W_0^{1,p}(\Omega;\Gamma_D)$. Let us notice that $\mathbb{W}_p(\Omega)$ equipped with the norm

$$\|y\|_{p,\nabla} := \|\nabla y\|_{L^p(\Omega)} = \left(\int_\Omega |\nabla y|^p\,dx\right)^{1/p} = \left(\int_\Omega \left|\sum_{i=1}^N \frac{\partial y}{\partial x_i}\right|^p dx\right)^{1/p} \tag{2.2}$$

is a uniformly convex Banach space [3]. Moreover, the norm $\|\cdot\|_{p,\nabla}$ is equivalent on $\mathbb{W}_p(\Omega)$ to the usual norm of $W^{1,p}(\Omega)$. By $BV(\Omega)$ we denote the space of all functions in $L^1(\Omega)$ for which the norm

$$\|f\|_{BV(\Omega)} = \|f\|_{L^1(\Omega)} + \int_\Omega |Df| = \|f\|_{L^1(\Omega)} \\ + \sup\Big\{ \int_\Omega f \operatorname{div}\varphi\, dx \ : \ \varphi \in C_0^1(\Omega;\mathbb{R}^N),\ |\varphi(x)| \le 1 \text{ for } x \in \Omega \Big\}$$

is finite.

We recall that a sequence $\{f_k\}_{k=1}^\infty$ converges weakly-$*$ to f in $BV(\Omega)$ if and only if the two following conditions hold (see [6]): $f_k \to f$ strongly in $L^1(\Omega)$ and $Df_k \rightharpoonup Df$ weakly-$*$ in the space of Radon measures $\mathcal{M}(\Omega)$, i.e.

$$\lim_{k\to\infty} \int_\Omega \varphi\, Df_k = \int_\Omega \varphi\, Df \quad \forall\, \varphi \in C_0(\Omega).$$

The following compactness result for BV-spaces is well-known (Helly's selection theorem, see [2]).

Theorem 2.1. *If $\{f_k\}_{k=1}^\infty \subset BV(\Omega)$ and $\sup_{k\in\mathbb{N}} \|f_k\|_{BV(\Omega)} < +\infty$, then there exists a subsequence of $\{f_k\}_{k=1}^\infty$ strongly converging in $L^1(\Omega)$ to some $f \in BV(\Omega)$ such that $Df_k \overset{*}{\rightharpoonup} Df$ in the space of Radon measures $\mathcal{M}(\Omega)$. Moreover, if $\{f_k\}_{k=1}^\infty \subset BV(\Omega)$ strongly converges to some f in $L^1(\Omega)$ and satisfies condition $\sup_{k\in\mathbb{N}} \int_\Omega |Df_k| < +\infty$, then*

$$\begin{aligned} &(i)\ f \in BV(\Omega) \quad \textit{and} \quad \int_\Omega |Df| \le \liminf_{k\to\infty} \int_\Omega |Df_k|; \\ &(ii)\ f_k \overset{*}{\rightharpoonup} f \quad \textit{in} \quad BV(\Omega). \end{aligned} \tag{2.3}$$

3. Setting of the Optimal Control Problem

Let ξ_1, ξ_2 be fixed elements of $L^\infty(\Omega) \cap BV(\Omega)$ satisfying the conditions

$$0 < \alpha \le \xi_1(x) \le \xi_2(x) \text{ a.e. in } \Omega, \tag{3.1}$$

where α is a given positive value.

Let $F : \Omega \times \mathbb{R} \to \mathbb{R}$ be a nonlinear mapping such that F is in the space $\operatorname{Car}(\Omega \times \mathbb{R})$ of Carathéodory functions on $\Omega \times \mathbb{R}$, i.e.

— the function $F(x,\cdot)$ is continuous in $\mathbb{R}$ for almost all $x \in \Omega$;

— the function $F(\cdot,y)$ is measurable for each $y \in \mathbb{R}$.

In addition, the following conditions of subcritical growth, monotonicity, and non-negativity are fulfilled:

$$|F(x,\eta)| \le C_1|\eta|^{r-1} \quad \text{for a.e. } x \in \Omega \text{ and all } \eta \in \mathbb{R}, \tag{3.2}$$

$$\big(F(x,\eta) - F(x,\eta')\big)\,(\eta - \eta') > 0 \quad \text{for a.e. } x \in \Omega \text{ and all } \eta,\eta' \in \mathbb{R},\ \eta \ne \eta', \tag{3.3}$$

$$F(x,\eta)\eta \ge 0 \quad \text{for a.e. } x \in \Omega \text{ and all } \eta \in \mathbb{R}, \tag{3.4}$$

for some $r \in (1, p^*)$, where

$$p^* = \begin{cases} Np/(N-p), & p < N, \\ +\infty, & p \ge N \end{cases}$$

is the critical exponent for the Sobolev imbedding $W^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$, and $C_1 > 0$.

Let $f \in W^{-1,q}(\Omega;\Gamma_D)$, $y_d \in L^2(\Omega)$, and $\zeta^{max} \in L^p(\partial\Omega)$ be given distributions. The optimal control problem we consider in this paper is to minimize the discrepancy between y_d and the solutions of the following state-constrained boundary valued problem

$$-\Delta_p(u(x),y) + F(x,y) = f(x) \quad \text{in} \quad \Omega, \tag{3.5}$$

$$y = 0 \quad \text{on} \;\; \Gamma_D, \quad \frac{\partial y(s)}{\partial \nu} = 0 \quad \text{on } \Gamma_N, \tag{3.6}$$

$$0 \le y(s) \le \zeta^{max}(s) \quad \text{a.e. on} \;\; \Gamma_N \tag{3.7}$$

by choosing an appropriate weight function $u \in \mathfrak{A}_{ad}$ as control. Here,

$$\Delta_p(u,y) := \operatorname{div}(u|\nabla y|^{p-2}\nabla y)$$

is the operator of the second order called the generalized p-harmonic operator, and the class of admissible controls $\mathfrak{A}_{ad}$ we define as follows

$$\mathfrak{A}_{ad} = \Big\{ u \in L^1(\Omega) \;\Big|\; \xi_1(x) \le u(x) \le \xi_2(x) \text{ a.e. in } \Omega \Big\}. \tag{3.8}$$

It is clear that $\mathfrak{A}_{ad}$ is a nonempty convex subset of $L^1(\Omega)$ with an empty topological interior.

More precisely, we are concerned with the following optimal control problem

$$\begin{gathered} \text{Minimize } \left\{ I(u,y) = \int_\Omega |y - y_d|^2\,dx + \int_\Omega |Du| \right\} \\ \text{subject to the constraints (3.5)–(3.8).} \end{gathered} \tag{3.9}$$

Before we will discuss the question of existence of admissible pairs to the problem (3.9), we note that the function $F \in \operatorname{Car}(\Omega \times \mathbb{R})$ can be associated with operator $\mathbf{F} : \mathbb{W}_p(\Omega) \to (\mathbb{W}_p(\Omega))^*$ defined by the rule

$$\langle \mathbf{F}(y), v \rangle_{(\mathbb{W}_p(\Omega))^*;\mathbb{W}_p(\Omega)} = \int_\Omega F(x,y)v\,dx, \quad \forall\, v \in \mathbb{W}_p(\Omega). \tag{3.10}$$

Moreover, taking into account the growth condition (3.2) and the compactness of the Sobolev imbedding $W_0^{1,p}(\Omega;\Gamma_D) \hookrightarrow L^r(\Omega)$ for $r < p^*$ it is easy to show that operator $\mathbf{F} : \mathbb{W}_p(\Omega) \to (\mathbb{W}_p(\Omega))^*$ is compact.

Definition 3.1. We say that an element $y \in \mathbb{W}_p(\Omega)$ is the weak solution (in the sense of Minty) to the boundary value problem (3.5)–(3.6), for a given admissible control $u \in \mathfrak{A}_{ad}$, if

$$\int_\Omega u|\nabla\varphi|^{p-2}\left(\nabla\varphi, \nabla\varphi - \nabla y\right)\,dx + \langle \mathbf{F}(\varphi), \varphi - y\rangle_{W^{-1,q}(\Omega;\Gamma_D);W_0^{1,p}(\Omega;\Gamma_D)}$$
$$\geq \langle f, \varphi - y\rangle_{W^{-1,q}(\Omega;\Gamma_D);W_0^{1,p}(\Omega;\Gamma_D)}, \quad \forall\,\varphi \in C_0^\infty(\Omega;\Gamma_D). \quad (3.11)$$

Remark 3.1. Since the integrand on the left hand side of inequality (3.11) has, in view of (3.10), the structure of a composite functions $g(x, y, \nabla y)$, it follows from Krasnosel'ski's well-know theorem (see [5] Ch.IV, Proposition 1.1) that $\varphi \mapsto g(x, y, \nabla y)$ is a continuous map $\mathbb{W}_p(\Omega) \to L^q(\Omega)$. Using the fact that the set $C_0^\infty(\mathbb{R}^N;\Gamma_D)$ is dense in $\mathbb{W}_p(\Omega)$, this allows us to take an arbitrary element $\varphi \in \mathbb{W}_p(\Omega)$ for a test function in (3.11). Therefore, taking $\varphi = y + tw$ with $w \in \mathbb{W}_p(\Omega)$ and $t > 0$, we obtain

$$\int_\Omega u|\nabla y + t\nabla w|^{p-2}\left(\nabla y + t\nabla w, \nabla w\right)\,dx + \int_\Omega F(x, y + tw)w\,dx$$
$$\geq \langle f, w\rangle_{W^{-1,q}(\Omega;\Gamma_D);W_0^{1,p}(\Omega;\Gamma_D)}, \quad \forall\,w \in \mathbb{W}_p(\Omega).$$

Passing to the limit as $t \to 0$ (because $F \in \mathrm{Car}\,(\Omega \times \mathbb{R})$), we get

$$\int_\Omega u|\nabla y|^{p-2}\left(\nabla y, \nabla w\right)\,dx + \int_\Omega F(x, y)w\,dx \geq \langle f, w\rangle_{W^{-1,q}(\Omega;\Gamma_D);W_0^{1,p}(\Omega;\Gamma_D)}$$

for all $w \in \mathbb{W}_p(\Omega)$. Hence,

$$\int_\Omega u|\nabla y|^{p-2}\left(\nabla y, \nabla w\right)\,dx + \int_\Omega F(x, y)w\,dx = \langle f, w\rangle_{W^{-1,q}(\Omega;\Gamma_D);W_0^{1,p}(\Omega;\Gamma_D)}. \quad (3.12)$$

It is worth to note that having applied Green's formula to the operator $-\operatorname{div}(u|\nabla y|^{p-2}\nabla y)$ tested by $v \in C_0^\infty(\Omega;\Gamma_D)$, we arrive at the identity

$$-\int_\Omega \operatorname{div}(u|\nabla y|^{p-2}\nabla y)v\,dx = \int_\Omega u|\nabla y|^{p-2}\left(\nabla y, \nabla v\right)\,dx$$
$$-\int_{\Gamma_N} u|\nabla y|^{p-2} v\frac{\partial y}{\partial \nu}\,d\mathcal{H}^{N-1} \quad \forall\,v \in C_0^\infty(\Omega;\Gamma_D).$$

Hence, if y as an element of $\mathbb{W}_p(\Omega) := W_0^{1,p}(\Omega;\Gamma_D)$ is the weak solution of the boundary value problem (3.5)–(3.6) in the sense of Definition 3.1, then relations (3.5)–(3.6) are fulfilled as follows

$$\left.\begin{array}{c} -\Delta_p(u, y) + \mathbf{F}(y) = f \quad \text{in} \quad \left(C_0^\infty(\Omega;\Gamma_D)\right)^*, \\ \gamma_0(y) = 0 \ \text{ in } \ W^{1/q,p}(\Gamma_D), \\ \gamma_1(y) = 0 \ \text{ in } \ W^{-1/p,p}(\Gamma_N). \end{array}\right\}$$

In particular, taking $w = y$ in (3.12), this yields the relation

$$\int_\Omega u|\nabla y|^p\,dx + \int_\Omega F(x,y)y\,dx = \langle f, y\rangle_{W^{-1,q}(\Omega;\Gamma_D);W_0^{1,p}(\Omega;\Gamma_D)}\,, \tag{3.13}$$

which is usually referred to as the energy equality. As a result, conditions (3.1), (3.8), and inequality (3.4) lead us to the following a priori estimate

$$\|y\|_{p,\nabla} := \left(\int_\Omega |\nabla y|^p\,dx\right)^{1/p} \le \left(\alpha^{-1}C_p\|f\|_{W^{-1,q}(\Omega;\Gamma_D)}\right)^{\frac{1}{p-1}} \quad \forall\, u \in \mathfrak{A}_{ad}. \tag{3.14}$$

The existence of a unique weak solution to the boundary value problem (3.5)–(3.6) in the sense of Definition 3.1 follows from an abstract theorem on monotone operators.

Theorem 3.1 ([9]). *Let V be a reflexive separable Banach space. Let V^* be the dual space, and let $A : V \to V^*$ be a bounded, hemicontinuous, coercive and strictly monotone operator. Then the equation $Ay = f$ has a unique solution for each $f \in V^*$.*

Here, the above mentioned properties of the strict monotonicity, hemicontinuity, and coercivity of the operator A have respectively the following meaning:

$$\langle Ay - Av, y - v\rangle_{V^*;V} \ge 0, \quad \forall\, y, v \in V; \tag{3.15}$$

$$\langle Ay - Av, y - v\rangle_{V^*;V} = 0 \implies y = v; \tag{3.16}$$

$$\text{the function } \; t \mapsto \langle A(y + tv), w\rangle_{V^*;V} \; \text{ is continuous for all } \; y, v, w \in V; \tag{3.17}$$

$$\lim_{\|y\|_V \to \infty} \frac{\langle Ay, y\rangle_{V^*;V}}{\|y\|_V} = +\infty. \tag{3.18}$$

In our case, we can define the operator $A(u, \cdot)$ as a mapping $\mathbb{W}_p(\Omega) \to (\mathbb{W}_p(\Omega))^*$ by

$$\langle A(u, y), w\rangle_{(\mathbb{W}_p(\Omega))^*;\mathbb{W}_p(\Omega)} := \int_\Omega u|\nabla y|^{p-2}\nabla y \nabla w\,dx + \int_\Omega F(x,y)w\,dx. \tag{3.19}$$

In view of the properties (3.2)–(3.4) and compactness of the Sobolev imbedding $W_0^{1,p}(\Omega;\Gamma_D) \hookrightarrow L^r(\Omega)$ for $r < p^*$, it is easy to show that $A(u, y) = -\Delta_p(u, y) + \mathbf{F}(y)$ and $A(u, \cdot)$ satisfies all assumptions of Theorem 3.1 (for the details we refer to [9, 11]). Hence, the variational problem

$$\begin{gathered}\text{For a given } u \in \mathfrak{A}_{ad}, \text{ find } y \in \mathbb{W}_p(\Omega) \text{ such that} \\ \langle A(u, y), \varphi\rangle_{(\mathbb{W}_p(\Omega))^*;\mathbb{W}_p(\Omega)} = \langle f, \varphi\rangle_{(\mathbb{W}_p(\Omega))^*;\mathbb{W}_p(\Omega)}\,, \; \forall\, \varphi \in \mathbb{W}_p(\Omega)\end{gathered} \tag{3.20}$$

for which $A(u, y) = f$ is its operator form, has a unique solution $y = y(u) \in \mathbb{W}_p(\Omega)$. We note that the duality pairing in the right hand side of (3.20) makes a sense for any distribution $f \in W^{-1,q}(\Omega;\Gamma_D)$ because

$$W^{-1,q}(\Omega;\Gamma_D) := \left(W_0^{1,p}(\Omega;\Gamma_D)\right)^*.$$

It remains to show that the solution y of (3.20) satisfies the Minty relation (3.11). Indeed, in view of the monotonicity of A, we have

$$\begin{aligned}
0 &\leq \langle A(u,v) - A(u,y), v - y\rangle_{(\mathbb{W}_p(\Omega))^*;\mathbb{W}_p(\Omega)} \\
&= \langle A(u,v), v - y\rangle_{(\mathbb{W}_p(\Omega))^*;\mathbb{W}_p(\Omega)} - \langle A(u,y), v - y\rangle_{(\mathbb{W}_p(\Omega))^*;\mathbb{W}_p(\Omega)} \\
&\overset{\text{by (3.20)}}{=} \langle A(u,v), v - y\rangle_{(\mathbb{W}_p(\Omega))^*;\mathbb{W}_p(\Omega)} - \langle f, \varphi\rangle_{W^{-1,q}(\Omega;\Gamma_D);W_0^{1,p}(\Omega;\Gamma_D)}.
\end{aligned}$$

Thus,

$$\langle A(u,v), v - y\rangle_{(\mathbb{W}_p(\Omega))^*;\mathbb{W}_p(\Omega)} \geq \langle f, \varphi\rangle_{W^{-1,q}(\Omega;\Gamma_D);W_0^{1,p}(\Omega;\Gamma_D)}, \quad \forall\, v \in W_0^{1,p}(\Omega;\Gamma_D),$$

and, hence, in view of Remark 3.1, the Minty relation (3.11) holds true.

Taking this fact into account, we adopt the following notion.

Definition 3.2. We say that (u,y) is an admissible pair to the OCP (3.9) if $u \in \mathfrak{A}_{ad} \subset L^1(\Omega)$, $y \in \mathbb{W}_p(\Omega)$, the pair (u,y) is related by the Minty inequality (3.11), $I(u,y) < +\infty$, and

$$\gamma_0(y) \in L^p_+(\Gamma_N), \quad \zeta^{max} - \gamma_0(y) \in L^p_+(\Gamma_N), \tag{3.21}$$

where $L^p_+(\Gamma_N)$ stands for the natural ordering cone of positive elements in $L^p(\Gamma_N)$, i.e.

$$L^p_+(\Gamma_N) := \left\{ v \in L^p(\Gamma_N) \mid v \geq 0 \quad \mathcal{H}^{N-1}\text{-a.e. on } \ \Gamma_N \right\}.$$

We denote by Ξ the set of all admissible pairs for the OCP (3.9). Let τ be the topology on the set $\Xi \subset L^1(\Omega) \times \mathbb{W}_p(\Omega)$ which we define as the product of the norm topology of $L^1(\Omega)$ and the weak topology of $W_0^{1,p}(\Omega;\Gamma_D)$. We say that a pair $(u^0, y^0) \in L^1(\Omega) \times \mathbb{W}_p(\Omega)$ is an optimal solution to problem (3.9) if

$$(u^0, y^0) \in \Xi \quad \text{and} \quad I(u^0, y^0) = \inf_{(u,y)\in\Xi} I(u,y).$$

Remark 3.2. Before we proceed further, we need to make sure that minimization problem (3.9) is meaningful, i.e. there exists at least one pair (u,y) such that (u,y) satisfying the control and state constraints (3.6)–(3.8), $I(u,y) < +\infty$, and (u,y) would be a physically relevant solution to the boundary value problem (3.5)–(3.6)? In fact, one needs the set of admissible solutions to be nonempty. But even if we are aware that $\Xi \neq \emptyset$, this set must be sufficiently rich in some sense, otherwise the OCP (3.9) becomes trivial. From a mathematical point of view, to deal directly with the control and especially state constraints is typically very difficult [4, 8, 12]. Thus, the regularity of OCPs with control and state constraints is an open question even for the simplest situation.

It is reasonably now to make use of the following Hypothesis.

(H_1) OCP (3.9) is regular in the following sense — there exists at least one pair $(u,y) \in L^1(\Omega) \times \mathbb{W}_p(\Omega)$ such that $(u,y) \in \Xi$.

4. Existence of Optimal Solutions

In this section we focus on the solvability of optimal control problem (3.5)–(3.9). Hereinafter, we suppose that the space $L^1(\Omega) \times \mathbb{W}_p(\Omega)$ is endowed with the norm $\|(u,y)\|_{L^1(\Omega)\times\mathbb{W}_p(\Omega)} := \|u\|_{L^1(\Omega)} + \|y\|_{p,\nabla}$.

We begin with a couple of auxiliary results.

Lemma 4.1. *Let $\{(u_k, y_k) \in \Xi\}_{k\in\mathbb{N}}$ be a sequence such that $(u_k, y_k) \xrightarrow{\tau} (u,y)$ in $L^1(\Omega) \times \mathbb{W}_p(\Omega)$. Then we have*

$$\lim_{k\to\infty} \int_\Omega u_k \left(\nabla y_k, \nabla\varphi\right)\, dx = \int_\Omega u \left(\nabla y, \nabla \varphi\right)\, dx \qquad \forall \varphi \in C_0^\infty(\Omega;\Gamma_D). \tag{4.1}$$

Proof. Since $u_k \to u$ in $L^1(\Omega)$ and $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$, we get that $u_k \to u$ strongly in $L^r(\Omega)$ for every $1 \le r < +\infty$. In particular, we have that $u_k \to u$ in $L^q(\Omega)$ and $(\nabla y_k, \nabla\varphi) \rightharpoonup (\nabla y, \nabla\varphi)$ in $L^p(\Omega)$. Hence, it is immediate to pass to the limit and to deduce (4.1). □

As a consequence, we have the following property.

Corollary 4.1. *Let $\{(u_k, y_k) \in \Xi\}_{k\in\mathbb{N}}$ and $\left\{\zeta_k \in W_0^{1,q}(\Omega;\Gamma_D)\right\}_{k\in\mathbb{N}}$ be sequences such that $(u_k, y_k) \xrightarrow{\tau} (u,y)$ in $L^1(\Omega)\times\mathbb{W}_p(\Omega)$ and $\zeta_k \to \zeta$ in $W_0^{1,q}(\Omega;\Gamma_D)$. Then*

$$\lim_{k\to\infty} \int_\Omega u_k \left(\nabla y_k, \nabla \zeta_k\right)\, dx = \int_\Omega u \left(\nabla y, \nabla\zeta\right)\, dx.$$

Our next step concerns the study of topological properties of the set of admissible solutions Ξ to problem (3.9).

The following result is crucial for our further analysis.

Theorem 4.1. *Let $\{(u_k, y_k)\}_{k\in\mathbb{N}} \subset \Xi$ be a bounded sequence in $BV(\Omega)\times\mathbb{W}_p(\Omega)$. Then there is a pair $(u,y) \in L^1(\Omega)\times\mathbb{W}_p(\Omega)$ such that, up to a subsequence, $(u_k, y_k) \xrightarrow{\tau} (u,y)$ and $(u,y)\in\Xi$.*

Proof. By Theorem 2.1 and compactness properties of the space $\mathbb{W}_p(\Omega)$, there exists a subsequence of $\{(u_k, y_k) \in \Xi\}_{k\in\mathbb{N}}$, still denoted by the same indices, and functions $u \in BV(\Omega)$ and $y \in \mathbb{W}_p(\Omega)$ such that

$$u_k \to u \text{ in } L^1(\Omega), \quad y_k \rightharpoonup y \;\text{ in }\; W_0^{1,p}(\Omega;\Gamma_D). \tag{4.2}$$

Then by Lemma 4.1, we have

$$\lim_{k\to\infty} \int_\Omega u_k \left(\nabla\varphi, \nabla y_k\right)\, dx = \int_\Omega u\left(\nabla\varphi, \nabla y\right)\, dx, \quad \forall\, \varphi \in C_0^\infty(\Omega;\Gamma_D).$$

It remains to show that the limit pair (u,y) is related by inequality (3.11) and satisfies the state constraints (3.21). With that in mind we write down the Minty relation for (u_k, y_k):

$$\int_\Omega u_k |\nabla\varphi|^{p-2} \left(\nabla\varphi, \nabla\varphi - \nabla y_k\right) dx + \int_\Omega F(x,\varphi)(\varphi - y_k)\, dx \\ \geq \langle f, \varphi - y_k\rangle_{W^{-1,q}(\Omega;\Gamma_D);W_0^{1,p}(\Omega;\Gamma_D)}, \quad \forall\, \varphi \in C_0^\infty(\Omega;\Gamma_D). \quad (4.3)$$

In view of (4.2) and Lemma 4.1, we have

$$\lim_{k\to\infty} \int_\Omega |\nabla\varphi|^p u_k\, dx = \int_\Omega |\nabla\varphi|^p u\, dx,$$
$$\lim_{k\to\infty} \int_\Omega u_k |\nabla\varphi|^{p-2} \left(\nabla\varphi, \nabla y_k\right) dx = \int_\Omega u |\nabla\varphi|^{p-2} \left(\nabla\varphi, \nabla y\right) dx.$$

Moreover, due to the compactness of the Sobolev imbedding $W_0^{1,p}(\Omega;\Gamma_D) \hookrightarrow L^r(\Omega)$ for $r < p^*$, we have

$$\int_\Omega F(x,\varphi)(\varphi - y_k)\, dx = \int_\Omega F(x,\varphi)(\varphi - y)\, dx + J_k,$$

where Hölder's inequality yields as $k \to \infty$

$$|J_k| := \left|\int_\Omega F(x,\varphi)(y - y_k)\, dx\right| \overset{\text{by (3.2)}}{\leq} \left(C_1 \int_\Omega |\varphi|^r\, dx\right)^{\frac{r-1}{r}} \|y - y_k\|_{L^r(\Omega)} \longrightarrow 0.$$

We, thus, can pass to the limit in relation (4.3) as $k \to \infty$ and arrive at the inequality (3.11), which means that $y \in \mathbb{W}_p(\Omega)$ is a weak solution to the boundary value problem (3.5)–(3.6) in the sense of Minty. Since the injections (2.1) are compact and the cone $L_+^p(\Gamma_N)$ is closed with respect to the strong convergence in $L^p(\Gamma_N)$, it follows that $y_k \to y$ strongly in $L^p(\Gamma_N)$ and, hence,

$$\lim_{k\to\infty} \gamma_0(y_k) = \gamma_0(y) \in L_+^p(\Gamma_N) \quad \text{and} \quad \gamma_0(y) \in \zeta^{max} - L_+^p(\Gamma_N).$$

This fact together with $u \in \mathfrak{A}_{ad}$ leads us to the conclusion: $(u, y) \in \Xi$, i.e. the limit pair (u, y) is admissible to optimal control problem (3.9). The proof is complete. □

Remark 4.1. Having applied the arguments of Remark 3.1, it is easy to show that in this case the energy equality (3.13) holds true for every τ-cluster pair (u, y) mentioned in Theorem 4.1.

In conclusion of this section, we give the existence result for optimal pairs to problem (3.9).

Theorem 4.2. *Assume that, for given distributions $f \in W^{-1,q}(\Omega;\Gamma_D)$, $y_d \in L^2(\Omega)$, and $\zeta^{max} \in L^p(\partial\Omega)$, the Hypothesis (H_1) is valid. Then optimal control problem* (3.9) *admits at least one solution $(u^{opt}, y^{opt}) \in BV(\Omega) \times \mathbb{W}_p(\Omega)$.*

Proof. Since the set of admissible pairs Ξ is nonempty and the cost functional is bounded from below on Ξ, it follows that there exists a minimizing sequence $\{(u_k, y_k) \in \Xi\}_{k\in\mathbb{N}}$ to problem (3.9). Then the inequality

$$\inf_{(u,y)\in\Xi} I(u,y) = \lim_{k\to\infty}\left[\int_\Omega |y_k(x) - y_d(x)|^2\,dx + \int_\Omega |Du_k|\right] < +\infty,$$

implies the existence of a constant $C > 0$ such that

$$\sup_{k\in\mathbb{N}}\int_\Omega |Du_k| \le C.$$

Hence, in view of the definition of the class of admissible controls $\mathfrak{A}_{ad}$ and a priori estimate (3.14), the sequence $\{(u_k, y_k) \in \Xi\}_{k\in\mathbb{N}}$ is bounded in $BV(\Omega) \times \mathbb{W}_p(\Omega)$. Therefore, by Theorem 4.1, there exist functions $u^* \in \mathfrak{A}_{ad}$ and $y^* \in \mathbb{W}_p(\Omega)$ such that $(u^*, y^*) \in \Xi$ and, up to a subsequence, $u_k \to u^*$ strongly in $L^1(\Omega)$ and $y_k \rightharpoonup y^*$ weakly in $W_0^{1,p}(\Omega;\Gamma_D)$. To conclude the proof, it is enough to show that the cost functional I is lower semicontinuous with respect to the *tau*-convergence. Since $y_k \to y^*$ strongly in $L^p(\Omega)$ by Sobolev embedding theorem, it follows that

$$\lim_{k\to\infty}\int_\Omega |y_k(x) - y_d(x)|^2\,dx = \int_\Omega |y^*(x) - y_d(x)|^2\,dx \quad \text{and,}$$

$$\liminf_{k\to\infty}\int_\Omega |Du_k| \ge \int_\Omega |Du^*| \text{ by (2.3).}$$

Thus,

$$I(u^*, y^*) \le \liminf_{k\to\infty} I(u_k, y_k) = \inf_{(u,\,y)\in\,\Xi} I(u,y).$$

Hence, (u^*, y^*) is an optimal pair, and we arrive at the required conclusion. □

5. Henig Relaxation of State-Constrainted OCP

As was mentioned above, the pointwise inequality constraints

$$0 \le y(s) \le \zeta^{max}(s) \quad \text{a.e. on } \Gamma_N$$

can be equivalently rewritten as $\gamma_0(y) \in L^p_+(\Gamma_N)$ and $\zeta^{max} - \gamma_0(y) \in L^p_+(\Gamma_N)$, where $L^p_+(\Gamma_N)$ stands for the natural ordering cone of positive elements in $L^p(\Gamma_N)$. From practical point of view it means that we cannot apply to OCP (3.9) any constraint qualifications like the Slater condition or the Robinson condition because each of those approaches is essentially based on non-emptiness of the interiors of the ordering cone $L^p_+(\Gamma_N)$. However, in our case we have $\operatorname{int}(L^p_+(\Gamma_N)) = \emptyset$. Therefore, the main goal of this section is to provide a regularization of the pointwise state constraints by replacing the ordering cone $\Lambda := L^p_+(\Gamma_N)$ by its solid Henig approximation $(\Lambda)_\varepsilon$ (see [14]) and show that admissibility and solvability of OCP (3.9) can be characterized by solving the corresponding Henig relaxed problems in the limit as $\varepsilon \to 0$.

We begin with some formal descriptions and abstract results. Let Z be a real normed space, and let $\Lambda \subset Z$ be a closed ordering cone in Z.

Definition 5.1. A nonempty convex subset B of a nontrivial ordering cone $\Lambda \subset Z$ (i.e. $\Lambda \neq \{0_Z\}$, where 0_Z is the zero element in Z) is called *base* of Λ if for each element $z \in \Lambda \backslash \{0_Z\}$ there is a unique representation $z = \mu b$ where $\mu > 0$ and $b \in B$.

In what follows, we always assume that the ordering cone Λ has a closed base $B \subset \Lambda$. We note that, in general, bases are not unique. We denote the norm of Z by $\|\cdot\|_Z$, and for arbitrary elements $z_1, z_2 \in Z$ we define

$$z_1 \leq_\Lambda z_2 \;\Leftrightarrow\; z_2 - z_1 \in \Lambda \;\text{ as well as }\; z_1 <_\Lambda z_2 \;\Leftrightarrow\; z_2 - z_1 \in \Lambda \backslash \{0_Z\}\,.$$

In order to introduce a representation for a base of Λ, let Z^* be the topological dual space of Z, and let $\langle \cdot, \cdot \rangle_{Z^*,Z}$ be the dual pairing. Moreover, by

$$\Lambda^* := \left\{ z^* \in Z^* \,\middle|\, \langle z^*, z \rangle_{Z^*,Z} \geq 0 \;\; \forall\, z \in \Lambda \right\}$$

and

$$\Lambda^{\#} := \left\{ z^* \in Z^* \,\middle|\, \langle z^*, z \rangle_{Z^*,Z} > 0 \;\; \forall\, z \in \Lambda \backslash \{0_Z\} \right\}$$

we define the *dual cone* and the *quasi-interior of the dual cone* of Λ, respectively. Using the definition of the dual cone, the ordering cone Λ can be characterized as follows (see [7, Lemma 3.21]):

$$\Lambda = \left\{ z \in Z \,\middle|\, \langle z^*, z \rangle_{Z^*,Z} \geq 0 \;\; \forall\, z^* \in \Lambda^* \right\}$$

Due to Lemma 1.28 in [7], we can give the following result.

Lemma 5.1. *Let $\Lambda \subset Z$ be a nontrivial ordering cone in a Banach space Z. Then the set $B := \{z \in \Lambda \,|\, \langle z^*, z \rangle_{Z^*,Z} = 1\}$ is a base of Λ for every $z^* \in \Lambda^{\#}$. Moreover, if Λ is reproducing in Z, i.e. if $\Lambda - \Lambda = Z$, and if B is a base of Λ, then there is an element $z^* \in \Lambda^{\#}$ satisfying $B = \{z \in \Lambda \,|\, \langle z^*, z \rangle_{Z^*,Z} = 1\}$.*

Remark 5.1. As follows from Lemma 5.1, the set

$$B := \left\{ \xi \in L^p_+(\Gamma_N) \,\middle|\, \int_{\Gamma_N} \xi \, d\mathcal{H}^{N-1} = 1 \right\} \tag{5.1}$$

is a closed base of ordering cone $\Lambda := L^p_+(\Gamma_N)$.

Now, we are prepared to introduce the definition of a so-called *Henig dilating cone* (see Zhuang, [14]) which is based on the existence of a closed base of ordering cone Λ.

Definition 5.2. Let Z be a normed space, and let $\Lambda \subset Z$ be a closed ordering cone with a closed base B. Choosing $\varepsilon > 0$ arbitrarily, the corresponding *Henig dilating cone* is defined by

$$\Lambda_\varepsilon(B) := \mathrm{cl}_{\|\cdot\|_Z}\Big(\mathrm{cone}\left(B + B_\varepsilon(0_Z)\right)\Big) := \mathrm{cl}_{\|\cdot\|_Z}\Big(\left\{\mu z \,\middle|\, \mu \geq 0, z \in B + B_\varepsilon(0_Z)\right\}\Big),$$

where $\frac{1}{\varepsilon} B_\varepsilon(0_Z) := \{v \in Z \mid \|v\|_Z \leq 1\}$ is the closed unit ball in Z centered at the origin.

It is clear that $\Lambda_\varepsilon(B)$ depends on the particular choice of B. As follows from this definition, $\operatorname{int}(\Lambda)_\varepsilon(B) \neq \emptyset$ for every $\varepsilon > 0$, i.e. Henig dilating cone is proper solid. Moreover, we have the following properties of such cones (see [13, 14]).

Proposition 5.1. Let Z be a normed space, and let $\Lambda \subset Z$ be a closed ordering cone with a closed base B. Choosing $\varepsilon \in (0, \delta)$, where

$$\delta := \inf \left\{ \|b\|_Z \,\middle|\, b \in B \right\} > 0, \tag{5.2}$$

the following statements hold true.

(i) $\Lambda_\varepsilon(B)$ is pointed, i.e. $\Lambda_\varepsilon(B) \cap \big(-\Lambda_\varepsilon(B)\big) = \{0_Z\}$;

(ii) $\Lambda_\varepsilon(B) \subset \Lambda_{\varepsilon+\gamma}(B) \;\; \forall\, \gamma > 0$;

(iii) $\Lambda_\varepsilon(B) = \operatorname{cone}\Big(\operatorname{cl}_{\|\cdot\|_Z} \left(B + B_\varepsilon(0_Z)\right)\Big)$;

(iv) $\Lambda = \bigcap\limits_{0<\varepsilon<\delta} \Lambda_\varepsilon(B)$;

(v) the implication

$$\xi \in (\Lambda)_\varepsilon(B) \quad \Longrightarrow \quad \frac{\varepsilon}{\kappa+\varepsilon}\|\xi\|_Z + \xi \notin (-\Lambda)\,, \tag{5.3}$$
$$\text{i. e. } \; \xi \not\leq_\Lambda -\frac{\varepsilon}{\kappa+\varepsilon}\|\xi\|_Z$$

holds true with $\kappa = \sup\{\|\zeta\|_Z \; : \; \zeta \in B\}$.

In the context of constraint qualifications problem, the following result plays an important role.

Proposition 5.2. Let Z be a normed space, and let $\Lambda \subset Z$ be a closed ordering cone with a closed base B. Choosing $\varepsilon \in (0, \delta)$ arbitrarily, where δ is defined by (5.2), the inclusion

$$\Lambda \subset \{0_Z\} \cup \operatorname{int}\big(\Lambda_\varepsilon(B)\big) \tag{5.4}$$

holds true.

Proof. Let $z \in \Lambda \setminus \{0_Z\}$ be chosen arbitrarily. By the definition of a base there is a unique representation $z = \lambda b$ with $\lambda > 0$ and $b \in B$. Obviously,

$$z \in \operatorname{int}\big(\{\lambda b\} + B_{\lambda\varepsilon}\,(0_Z)\,\big) = \operatorname{int}\big(B_{\lambda\varepsilon}(\lambda b)\big)$$

holds true. Let's assume for a moment that

$$B_{\lambda\varepsilon}(\lambda b) \subseteq \operatorname{cone}\big(\{b\} + B_\varepsilon\,(0_Z)\,\big). \tag{5.5}$$

Then we obtain

$$z \in \operatorname{int}\Big(\operatorname{cone}\big(\{b\} + B_\varepsilon(0_Z)\big)\Big) \subseteq \operatorname{int}\Big(\operatorname{cone}\big(B + B_\varepsilon(0_Z)\big)\Big) = \operatorname{int}\big(\Lambda_\varepsilon(B)\big),$$

which completes the proof. In order to show (5.5), let $x \in B_{\lambda\varepsilon}(\lambda b)$ be chosen arbitrarily, i.e.

$$\|x - \lambda b\|_Z \le \lambda\varepsilon.$$

Then

$$\left\|\frac{x}{\lambda} - b\right\|_Z = \frac{1}{\lambda}\|x - \lambda b\|_Z \le \frac{\lambda\varepsilon}{\lambda} = \varepsilon$$

yields

$$x \in \left\{\mu y \,\middle|\, \|y - b\|_Z \le \varepsilon, \mu \ge 0\right\} = \operatorname{cone}\big(\{b\} + B_\varepsilon(0_Z)\big).$$

As a result, (5.5) is satisfied. □

Remark 5.2. The following property, coming from Proposition 5.2, turns out rather useful: in order to prove $z \in \operatorname{int}\big(\Lambda_\varepsilon(B)\big)$, it is sufficient to check whether $z \in \Lambda \setminus \{0_Z\}$.

The following result shows that Henig dilating cones $\Lambda_\varepsilon(B)$ possess good approximation properties.

Proposition 5.3. Let Λ be a closed ordering cone in a normed space Z, and let B be an arbitrary closed base of Λ. Let parameter δ be defined as in (5.2), and let $(\varepsilon_k)_{k\in\mathbb{N}} \subset (0,\delta)$ be a monotonically decreasing sequence such that $\lim\limits_{k\to\infty} \varepsilon_k = 0$. Then the sequence of cones $\{\Lambda_{\varepsilon_k}(B)\}_{k\in\mathbb{N}}$ converges to Λ in Kuratowski sense with respect to the norm topology of Z as k tends to infinity, that is

$$K\text{-}\liminf_{k\to\infty} \Lambda_{\varepsilon_k}(B) = \Lambda = K\text{-}\limsup_{k\to\infty} \Lambda_{\varepsilon_k}(B),$$

where

$$\begin{aligned} K\text{-}\liminf_{k\to\infty} \Lambda_{\varepsilon_k}(B) := \Big\{z \in Z \,\Big|\, &\text{ for all neighborhoods } N \text{ of } z \text{ there is a} \\ &k_0 \in \mathbb{N} \text{ such that } N \cap \Lambda_{\varepsilon_k}(B) \ne \emptyset \ \forall\, k \ge k_0\Big\}, \\ K\text{-}\limsup_{k\to\infty} \Lambda_{\varepsilon_k}(B) := \Big\{z \in Z \,\Big|\, &\text{ for all neighborhoods } N \text{ of } z \text{ and every } k_0 \in \mathbb{N} \\ &\text{there is a } k \ge k_0 \text{ such that } N \cap \Lambda_{\varepsilon_k}(B) \ne \emptyset\Big\}. \end{aligned}$$

Proof. Let $z \in \Lambda$ be chosen arbitrarily. Then $N \cap \Lambda \ne \emptyset$ holds true for every neighborhood N of z, and due to the inclusions $\Lambda \subset \Lambda_{\varepsilon_k}$ $\forall\, k \in \mathbb{N}$, we see that $N \cap \Lambda_{\varepsilon_k} \ne \emptyset$ for all $k \in \mathbb{N}$. Hence,

$$\Lambda \subseteq K\text{-}\liminf_{k\to\infty} \Lambda_{\varepsilon_k}(B). \tag{5.6}$$

Taking into account the inclusion (5.6) and the fact that

$$K\text{-}\liminf_{k\to\infty} \Lambda_{\varepsilon_k}(B) \subseteq K\text{-}\limsup_{k\to\infty} \Lambda_{\varepsilon_k}(B),$$

we get

$$\Lambda \subseteq K\text{-}\liminf_{k\to\infty} \Lambda_{\varepsilon_k}(B) \subseteq K\text{-}\limsup_{k\to\infty} \Lambda_{\varepsilon_k}(B). \tag{5.7}$$

To show that the sequence $\{\Lambda_{\varepsilon_k}(B)\}_{k\in\mathbb{N}}$ converges to Λ in Kuratowski sense, it remains to show

$$K-\limsup_{k\to\infty}\Lambda_{\varepsilon_k}(B)\subseteq\Lambda. \tag{5.8}$$

However, the inclusion (5.8) is equivalent to

$$(Z\backslash\Lambda)\subseteq\left(Z\Big\backslash K-\limsup_{k\to\infty}\Lambda_{\varepsilon_k}(B)\right). \tag{5.9}$$

Let $\bar z\in Z\backslash\Lambda$ be an arbitrarily element. Since Λ is closed, there is an open neighborhood $\bar N$ of $\bar z$ with respect to the norm topology of Z such that $\bar N\cap\Lambda=\emptyset$. By Proposition 5.1 (see item (iv)), there is a sufficiently large index $k_0\in\mathbb{N}$ such that

$$\bar N\cap\Lambda_{\varepsilon_k}(B)=\emptyset\ \ \forall\,k\geq k_0.$$

This implies

$$\bar z\in Z\Big\backslash\limsup_{k\to\infty}\Lambda_{\varepsilon_k}(B).$$

Combining (5.7), (5.8), and (5.9), we arrive at the relation

$$\Lambda\subseteq K-\liminf_{k\to\infty}\Lambda_{\varepsilon_k}(B)\subseteq K-\limsup_{k\to\infty}\Lambda_{\varepsilon_k}(B)\subseteq\Lambda.$$

Thus, $\Lambda=K-\lim\limits_{k\to\infty}\Lambda_{\varepsilon_k}(B)$ and the proof is complete. □

Taking these results into account, we associate with OCP (3.9) the following Henig relaxed problem

$$\text{Minimize }\left\{I(u,y)=\int_\Omega|y-y_d|^2\,dx+\int_\Omega|Du|\right\} \tag{5.10}$$

subject to the constraints

$$\left.\begin{array}{rcl}-\Delta_p(u,y)+\mathbf{F}(y)=f & \text{in} & \left(C_0^\infty(\Omega;\Gamma_D)\right)^*,\\ \gamma_0(y)=0 & \text{in} & W^{1/q,p}(\Gamma_D),\\ \gamma_1(y)=0 & \text{in} & W^{-1/p,p}(\Gamma_N),\\ \gamma_0(y) & \in & \left(L^p_+(\Gamma_N)\right)_\varepsilon(B),\\ \zeta^{max}-\gamma_0(y) & \in & \left(L^p_+(\Gamma_N)\right)_\varepsilon(B)\\ u & \in & \mathfrak{A}_{ad},\end{array}\right\} \tag{5.11}$$

or in a more compact form this problem can be stated as follows

$$\inf_{(u,y)\in\Xi_\varepsilon}I(u,y),\quad\forall\,\varepsilon\in(0,\delta) \tag{5.12}$$

where

$$\delta=\inf\left\{\|\xi\|_{L^p(\Gamma_N)}\ :\ \xi\in B\right\}, \tag{5.13}$$

the base B takes the form (5.1), and the set of admissible solutions $\Xi_\varepsilon \subset L^1(\Omega) \times \mathbb{W}_p(\Omega)$ we define as follows: $(u, y) \in \Xi_\varepsilon$ if and only if $u \in \mathfrak{A}_{ad}$, $I(u, y) < +\infty$, $y \in \mathbb{W}_p(\Omega)$, the pair (u, y) is related by the Minty inequality (3.11), and

$$\gamma_0(y) \in \left(L^p_+(\Gamma_N)\right)_\varepsilon (B), \quad \zeta^{max} - \gamma_0(y) \in \left(L^p_+(\Gamma_N)\right)_\varepsilon (B). \tag{5.14}$$

Here, $\left(L^p_+(\Gamma_N)\right)_\varepsilon (B)$ is the corresponding Henig dilating cone.

Since, by Proposition 5.2, the inclusion $\Xi \subseteq \Xi_\varepsilon$ holds true for all $\varepsilon > 0$, it is reasonable to call the OCP (5.12) a relaxation of OCP (3.9). Moreover, as obviously follows from Proposition 5.3, the convergence $\Xi_\varepsilon \overset{\varepsilon\to 0}{\to} \Xi$ in Kuratowski sense holds true with respect to the τ-topology on $L^1(\Omega) \times \mathbb{W}_p(\Omega)$.

We are now in a position to show that using the relaxation approach we can reduce the main suppositions of Theorem 4.2. In particular, we can characterize Hypothesis (H_1) by the regularity properties of the corresponding Henig relaxed problems.

Theorem 5.1. *Let $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0, \delta)$ be a monotonically decreasing sequence converging to 0 as $k \to \infty$. Then, for given distributions $f \in W^{-1,q}(\Omega; \Gamma_D)$, $y_d \in L^2(\Omega)$, and $\zeta^{max} \in L^p(\partial\Omega)$, the Hypothesis (H1) implies that the Henig relaxed problem (5.12) has a nonempty set of admissible solutions Ξ_ε for all $\varepsilon = \varepsilon_k$, $k \in \mathbb{N}$. And vice versa, if there exists a sequence $\left\{(u^k, y^k)\right\}_{k\in\mathbb{N}}$ satisfying conditions*

$$(u^k, y^k) \in \Xi_{\varepsilon_k} \ \ \textit{for all} \ \ k \in \mathbb{N}, \qquad \textit{and} \ \ \sup_{k\in\mathbb{N}} I(u^k, y^k) < +\infty, \tag{5.15}$$

then each of τ-cluster pairs of this sequence is admissible to the original OCP (3.9).

Proof. Since the implication $\big(\Xi \neq \emptyset\big) \implies \big(\Xi_\varepsilon \neq \emptyset$ for all $\ \varepsilon > 0\big)$ is obvious by Proposition 5.3, we concentrate on the proof of the inverse statement — regularity of the Henig relaxed problems $\inf_{(u,y)\in\Xi_{\varepsilon_k}} I(u, y)$ for all $k \in \mathbb{N}$ with property (5.15) implies the existence of at least one pair (u, y) such that $(u, y) \in \Xi$.

Let $\left\{(u^k, y^k)\right\}_{k\in\mathbb{N}}$ be an arbitrary sequence with property: $(u^k, y^k) \in \Xi_{\varepsilon_k}$ for all $k \in \mathbb{N}$. Since the set $\mathfrak{A}_{ad}$ and a priory estimate (3.14) do not depend on parameter ε_k and the condition (5.15) implies $\sup_{k\in\mathbb{N}} \int_\Omega |Du_k| \le \infty$, it follows by compactness arguments (see the proof of Theorem 4.2) that there exist a subsequence of $\left\{(u^k, y^k)\right\}_{k\in\mathbb{N}}$ (still denoted by the same index) and a pair $(u^*, y^*) \in \mathfrak{A}_{ad} \times \mathbb{W}_p(\Omega)$ such that

$$(u^k, y^k) \xrightarrow{\ \tau\ } (u^*, y^*) \ \text{ as } k \to \infty.$$

Closely following the proof of Theorem 4.1, it can be shown that the limit pair (u^*, y^*) is such that $u^* \in \mathfrak{A}_{ad}$, $J(u^*, y^*) < +\infty$, and function $y^* \in \mathbb{W}_p(\Omega)$ is a weak solution (in the sense of Minty) to the boundary value problem (3.5)–(3.6). Moreover, in view of the compactness properties of injections (2.1), we may suppose that

$$\gamma_0(y^k) \to \gamma_0(y) \quad \text{strongly in} \ \ L^p(\Gamma_N) \ \text{ as } k \to \infty. \tag{5.16}$$

It remains to establish the inclusions

$$\gamma_0(y^*) \in L^p_+(\Gamma_N), \quad \zeta^{max} - \gamma_0(y^*) \in L^p_+(\Gamma_N), \tag{5.17}$$

By contraposition, let us assume that $\xi^* := \zeta^{max} - \gamma_0(y^*) \in L^p(\Gamma_N) \setminus L^p_+(\Gamma_N)$. Since the cone $L^p_+(\Gamma_N)$ is closed, it follows that there is a neighborhood $\mathcal{N}(\xi^*)$ of ξ^* in $L^p(\Gamma_N)$ such that $\mathcal{N}(\xi^*) \cap L^p_+(\Gamma_N) = \emptyset$. Using the fact that

$$L^p_+(\Gamma_N) \subset \left(L^p_+(\Gamma_N)\right)_{\varepsilon_k}(B) \subseteq \left(L^p_+(\Gamma_N)\right)_{\varepsilon_l}(B), \quad \forall\, k \geq l,$$

by Proposition 5.3 and definition of the Kuratowski limit, it is easy to conclude the existence of an index $k_0 \in \mathbb{N}$ such that

$$\mathcal{N}(\xi^*) \cap \left(L^p_+(\Gamma_N)\right)_{\varepsilon_k}(B) = \emptyset, \quad \forall\, k \geq k_0. \tag{5.18}$$

However, in view of the strong convergence property (5.16), there is an index $k_1 \in \mathbb{N}$ satisfying

$$\xi^k \in \mathcal{N}(\xi^*), \quad \forall\, k \geq k_1. \tag{5.19}$$

Combining (5.18) and (5.19), we finally obtain

$$\xi^k = \zeta^{max} - \gamma_0(y^k) \in L^p(\Gamma_N) \setminus \left(L^p_+(\Gamma_N)\right)_{\varepsilon_k}(B), \quad \forall\, k \geq \max\{k_0, k_1\}.$$

This, however, is a contradiction to

$$\zeta^{max} - \gamma_0(y^k) \in L^p_+(\Gamma_N), \quad \forall\, k \in \mathbb{N}.$$

Thus, $\zeta^{max} - \gamma_0(y^*) \in L^p_+(\Gamma_N)$. In the same manner it can be shown that $\gamma_0(y^*) \in L^p_+(\Gamma_N)$. Hence, the pair (u^*, y^*) is admissible for OCP (3.9). $\square$

As an obvious consequence of this theorem and Theorem 4.2, we have the following noteworthy property of the Henig relaxed problems (5.12).

Corollary 5.1. *Let $f \in W^{-1,q}(\Omega; \Gamma_D)$, $y_d \in L^2(\Omega)$, and $\zeta^{max} \in L^p(\partial\Omega)$ be given distribution. Then the Henig relaxed problem* (5.12) *is solvable for each $\varepsilon \in (0, \delta)$ provided Hypothesis (H_1) is satisfied.*

The next result is crucial in this section. We show that optimal solutions for the original OCP (3.9) can be attained by solving the corresponding Henig relaxed problems (5.10)–(5.11).

Theorem 5.2. *Let $f \in W^{-1,q}(\Omega; \Gamma_D)$, $y_d \in L^2(\Omega)$, and $\zeta^{max} \in L^p(\partial\Omega)$ be given distributions. Let $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0, \delta)$ be a monotonically decreasing sequence such that $\varepsilon_k \to 0$ as $k \to \infty$, where $\delta > 0$ is defined by* (5.13)*. Let $\left\{(u^{k,0}, y^{k,0}) \in \Xi_{\varepsilon_k}\right\}_{k\in\mathbb{N}}$ be a sequence of optimal solutions to the Henig relaxed problems* (5.10)–(5.11) *such that*

$$\sup_{k\in\mathbb{N}} \|u^{k,0}\|_{BV(\Omega)} < +\infty. \tag{5.20}$$

Then there is a subsequence $\left\{(u^{k_i,0}, y^{k_i,0})\right\}_{i\in\mathbb{N}}$ *of* $\left\{(u^{k,0}, y^{k,0})\right\}_{k\in\mathbb{N}}$ *and a pair* (u^0, y^0) *such that*

$$(u^{k_i,0}, y^{k_i,0}) \xrightarrow{\tau} (u^0, y^0) \quad \text{as } i \to \infty, \tag{5.21}$$

$$(u^0, y^0) \in \Xi, \quad \text{and} \quad I(u^0, y^0) = \inf_{(u,y)\in\Xi} I(u, y). \tag{5.22}$$

Proof. In view of a priory estimate (3.14), the uniform boundedness of optimal controls with respect to BV-norm (5.20) implies the fulfilment of condition $(5.15)_2$. Hence, the compactness property (5.21) and the inclusion $(u^0, y^0) \in \Xi$ are a direct consequence of Theorem 5.1. It remains to show that the limit pair (u^0, y^0) is a solution to OCP (3.9). Indeed, the condition $(u^0, y^0) \in \Xi$ implies regularity of the original OCP (3.9). Hence, by Theorem 4.2, this problem has a nonempty set of solutions. Let (u^*, y^*) be one of them. Then the following inequality is obvious

$$I(u^*, y^*) \leq I(u^0, y^0). \tag{5.23}$$

On the other hand, by Proposition 5.1 (see property (iv)), we have $(u^*, y^*) \in \Xi_{\varepsilon_{k_i}}$ for every $i \in \mathbb{N}$. Since $\left\{(u^{k_i,0}, y^{k_i,0})\right\}_{i\in\mathbb{N}}$ are the solutions to the corresponding relaxed problems (5.12), it follows that

$$\inf_{(u,y)\in\Xi_{\varepsilon_{k_i}}} I(u, y) = I(u^{k_i,0}, y^{k_i,0}) \leq I(u^*, y^*), \quad \forall\, i \in \mathbb{N}. \tag{5.24}$$

As a result, taking into account the relations (5.23) and (5.24), and the lower semicontinuity property of the cost functional I with respect to the τ-convergence, we finally get

$$\begin{aligned}\inf_{(u,y)\in\Xi} I(u, y) = I(u^*, y^*) &\overset{\text{by (5.24)}}{\geq} \limsup_{i\to\infty} I(u^{k_i,0}, y^{k_i,0}) \\ &\geq \liminf_{i\to\infty} I(u^{k_i,0}, y^{k_i,0}) \geq I(u^0, y^0) \overset{\text{by (5.23)}}{\geq} I(u^*, y^*).\end{aligned}$$

Thus,

$$\inf_{(u,y)\in\Xi} I(u, y) = \lim_{i\to\infty} I(u^{k_i,0}, y^{k_i,0}) = I(u^0, y^0),$$

and we arrive at the desired property $(5.22)_2$. The proof is complete. □

Remark 5.3. It is worth to note that condition (5.20) can be omitted if the original OCP (3.9) is regular, that is when Hypothesis (H_1) is valid. Indeed, let us assume that $\Xi \neq \emptyset$ and $(\widehat{u}, \widehat{y}) \in \Xi$ is an arbitrary pair. Then $(\widehat{u}, \widehat{y})$ is admissible to each Henig relaxed problems (5.10)–(5.11), and, hence,

$$\inf_{(u,y)\in\Xi_{\varepsilon_k}} I(u, y) = I(u^{k,0}, y^{k,0}) \leq I(\widehat{u}, \widehat{y}), \quad \forall\, k \in \mathbb{N}. \tag{5.25}$$

Since, by Proposition 5.2, the inclusion $\Xi \subseteq \Xi_{\varepsilon_k}$ holds true for all $\varepsilon_k > 0$, and the sequence $\{\Xi_{\varepsilon_k}\}_{k\in\mathbb{N}}$ is monotone in the following sense (because of the property (ii) of Proposition 5.1)

$$\Xi_{\varepsilon_1} \supseteq \Xi_{\varepsilon_2} \supseteq \cdots \supseteq \Xi_{\varepsilon_k} \supseteq \cdots \supseteq \Xi \neq \emptyset,$$

it follows that

$$\inf_{(u,y)\in\Xi_{\varepsilon_1}} I(u,y) \leq \cdots \leq \inf_{(u,y)\in\Xi_{\varepsilon_k}} I(u,y) \leq \cdots \leq \inf_{(u,y)\in\Xi} I(u,y) \leq I(\widehat{u},\widehat{y}).$$

As a result, (5.25) leads to the estimate

$$\begin{aligned}\sup_{k\in\mathbb{N}} \|u^{k,0}\|_{BV(\Omega)} &\leq \sup_{k\in\mathbb{N}} \left[\int_\Omega |u^{k,0}|\,dx + \inf_{(u,y)\in\Xi_{\varepsilon_k}} I(u,y)\right] \\ &\leq \|\xi_2\|_{L^\infty(\Omega)}|\Omega| + I(\widehat{u},\widehat{y}) < +\infty.\end{aligned}$$

As was mentioned at the beginning of this section, the main benefit of the relaxed optimal control problems (5.10)–(5.11) comes from the fact that the Henig dilating cone $\left(L^p_+(\Gamma_N)\right)_\varepsilon(B)$ has a nonempty topological interior. Hence, it gives a possibility to apply the Slater condition or the Robinson condition in order to characterize the optimal solutions for the state constrained OCP (3.9). On the other hand, this approach provides nice convergence properties for the solutions of relaxed problems (5.10)–(5.11). However, as follows from Theorems 5.1 and 5.2 (see also Remark 5.3), the most restrictive assumption deals with the regularity of the relaxed problems (5.10)–(5.11) for all $\varepsilon \in (0,\delta)$. So, if we reject the Hypothesis (H_1), it becomes unclear, in general, whether the relaxed sets of admissible solutions Ξ_ε are nonempty for all $\varepsilon \searrow 0$. In this case it makes sense to provide further relaxation for each of Henig problems (5.10)–(5.11).

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