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SOME GENERALIZATIONS OF THE NONLOCAL TRANSFORMATIONS APPROACH

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Some generalizations of a method of nonlocal transformations are proposed: a connection of given equations via prolonged nonlocal transformations and finding of an adjoint solution to the solutions of initial equation are considered. A concept of nonlocal transformation with additional variables is introduced, developed and used for searching symmetries of differential equations. A problem of inversion of the nonlocal transformation with additional variables is investigated and in some cases solved. Several examples are presented. Derived technique is applied for construction of the algorithms and formulae of generation of solutions. The formulae derived are used for construction of exact solutions of some nonlinear equations.

Key words: nonlocal transformations, Lie classical symmetry, nonlocal symmetries, additional variables, formulae for generation of solutions, nonlinear superposition principle, Bäcklund transformations.

1. Introduction

Wide range of effective methods for study and solving the nonlinear partial differential equations are developed up today. Substantial part of them have in the base the fundamental idea of symmetry of DE, and, thus, to the group theoretical method of Lie [1–3] belongs. Such methods got numerous useful generalizations by the present moment. Among most important of them there are approaches based on study of conditional (nonclassical), weak symmetries [4–6] and nonlocal symmetries of differential equations [7–14]. We do not claim to give complete references for all the known results in this text. A description of the main results of development in the specified field can be found, e.g., in [9, 11–13].

However, in a great number of cases important for applications the information obtained by classical group-theoretical method and its generalizations is quite poor. Therefore a development other approaches that provide searching new symmetries and methods of solution these equations stay topical.

Here we note, besides the others, two important sources of creation of new approaches. First idea is very old and is based on use of additional variables, particularly, when the point transformations are using for integration of differential equations. Remind, for instance, the Bernoulli substitution $y(x) = u(x)v(x)$ for solving ODEs, the Fourier substitution $u(x,t) = g(x)v(t)$ in the method of separation of variables to PDEs or Lagrange's method of variation of arbitrary constants for construction of a general solution $y(x) = C_1(x)y_1(x)+C_2(x)y_2(x)$ to inhomogeneous second order (for example) ODE from two known partial solutions $y_1(x)$, $y_2(x)$ of appropriate homogeneous equation. One can continue a list of examples this type [15, 16].

Essentially other approach is based on the use of a nonlocal correspondence between two given equations. Well-known method of Bäcklund transformations (BT) to this approach belongs and, being powerful tool for study of nonlinear PDEs, is widely used nowadays. Particular cases of application this method to geometrical researches were considered by Bianki, Ribokur, Darboux. Later this method was studied and generalized in works of Bäcklund and others [7]. Renewal of interest to this method in 70th of past century resulted in active development the theory of integrability for nonlinear differential equations.

" Bäcklund transformations, ... too are transformations in which the (independent and dependent) variables as well as their derivatives are involved, but each of them makes sense, and is well defined, only for an associated special set of partial differential equations and its image under this transformation." [7]. According to [7] a BT connecting the surface elements (x^1, x^2, u, u_1, u_2) and (y^1, y^2, v, v_1, v_2) is determined as a set of four equations

$$\Lambda_i(x^1, x^2, u, u_1, u_2; y^1, y^2, v, v_1, v_2) = 0, \quad i = 1, ..., 4, \tag{1.1}$$

solving which with respect to u_1, u_2 and x^j one gets the explicit solutions

$$u_i = H_i(y^1, y^2, v, v_1, v_2; u), \quad i = 1, 2, \tag{1.2}$$

and

$$x^j = h^j(y^1, y^2, v, v_1, v_2; u), \quad j = 1, 2. \tag{1.3}$$

Subscripts of functions denote differentiation with respect to the corresponding arguments. In the case of two independent variables we also will use the special notation of the variables: $x_1 = x$, $x_2 = t$ and thus $u_t = \partial u/\partial t = \partial_t u$, $u_x = \partial u/\partial x = \partial_x u$.

The integrability condition for (1.2) is

$$\partial_{x^2} H_1 = \partial_{x^1} H_2. \tag{1.4}$$

Notice that it may be thought of as the zero curvature condition for suitable connections [7]. If (1.4) generates the relation

$$\partial_{x^2} H_1 - \partial_{x^1} H_2 \Big|_{F_1(y^1, y^2, v_{(k)})} \equiv 0,$$

where

$$F_1(y^1, y^2, v_{(k)}) = 0,$$

and the reversion of this procedure leads to the equation for u

$$F_0(x^1, x^2, u_{(k)}) = 0,$$

it is said that the Bäcklund transformation connects (establish a correspondence between) given equations and each of functions u and v have to obey the corresponding differential equation. Here a symbol $u_{(r)}$ denotes the tuple of derivatives of the function u from order zero up to order $r \leq k$.

An approach uniting the Bäcklund transformations and the theory of potentials was generalized by Wahlquist and Estabrook via introducing new auxiliary variables. It allows them to develop a method of *pseudopotentials* known also as a *method of prolongations of structures* [17–22]. In the case of two independent variables $x^1 = x$, $x^2 = t$ they supposed an existence other potentials forming the pseudopotential $V = (v^{\mathrm{I}}, v^{\mathrm{II}}, ..., v^{\mathrm{M}})$ such that a system

$$v^{\mathrm{J}}_x = \phi^{\mathrm{J}} t(x, t, u_{(r)}, V), \quad v^{\mathrm{J}}_t = -\phi^{\mathrm{J}} x(x, t, u_{(r)}, V), \quad \mathrm{J} = (\mathrm{I}, \mathrm{II}, ..., \mathrm{M}), \tag{1.5}$$

replaces (1.2) [22]. So auxiliary variables $(v^{\mathrm{II}}, v^{\mathrm{III}}, ..., v^{\mathrm{M}})$ may be thought of as additional dependent variables introduced into the structure of the BT operating in the space of two independent variables (x, t).

On the other hand a number of interesting results for nonlinear equations connected among themselves by the *nonlocal transformations* are obtained for today and the formulae generating solutions or nonlocal nonlinear superposition are constructed. This approach is intended for search of the nonlocal symmetries admitted by PDEs and is based on the nonlocal transformations technic [23,25,26]. A nonlocal transformation

$$\mathcal{T}: \quad x^i = h^i(y, v_{(k)}), \quad i = 1, \ldots, n, \quad u^K = H^K(y, v_{(k)}), \quad K = 1, \ldots, m$$

allows mapping a given initial (*source*) equation into another (*target*) equation. We will describe a basic concept of this method in the beginning of the following section. Note that formulae of nonlocal nonlinear superposition obtained may be understood also as an ABT with an additional variable satisfying the same equation [12].

The two approaches are entirely different. The main difference between the Bäcklund correspondence of two given equations (systems of equations) and a nonlocal transformation connecting the same equations, roughly speaking, is reduced to the using in the BT of the *conjugate* differential equations, depending on two types of the dependent variables, that guarantee an integrability of the complete system with respect to each family of variables.

Suppose we know only one subset of equations forming the BT system and connecting a given DEs. Then a lack of conjugate subset of differential equations forming whole Bäcklund transformation sometimes can be restored from the known one, being used as a nonlocal transformation connecting these equations.

For example, consider the Cole-Hopf substitution

$$u = -2\, \partial_x \ln |v|, \tag{1.6}$$

connecting the Burgers equation

$$u_t + u\ u_x - u_{xx} = 0 \tag{1.7}$$

and the linear heat equation

$$v_t - v_{xx} = 0. \tag{1.8}$$

We aim to construct the lacking conjugate part of the BT. Differentiating the Cole-Hopf substitution with respect to x we get

$$u_x + 2\ v^{-1}v_{xx} - 2\ v^{-2}{v_x}^2 = 0.$$

Substituting into this result $v_{xx} = v_t$ and $v_x = -2^{-1}u\,v$, we find an expression

$$u_x + 2\ v^{-1}v_t - 2^{-1}u^2 = 0.$$

It is well-known second (conjugate) equation in the BT, which connects two given equations [22].

Finite nonlocal transformations are effectively used for investigation of the nonlinear differential equations for a long time. See, e.g., publications [10–12, 27–29] devoted to various applications of these transformations.

The usage of auxiliary dependent variables for deriving helpful information about nonlinear PDEs is a widespread technique. One of immediate and obvious generalizations of the nonlocal transformations approach assumes the subsequent connection of equation F_1 by new nonlocal transformation $\mathcal{T}_1$ with new equation F_2, then F_2 by $\mathcal{T}_2$ with F_3 and so on..., and use of the properties of these auxiliary equations for information about symmetries and solutions of the initial equation [10–12].

Within the potential symmetries approach note some results based on introduction next potential variables applying some iterating technique when previous potential is known. Such results were obtained by Akhatov, Gasisov and Ibragimov [30] and were used in more recent paper of King [31] where a tree of nonlocally related PDEs were constructed. The similar multipotentialisation has allowed Euler construct the iterating-solution formulae for Krichever-Novikov equation and others in [29]. The method of construction of a tree of nonlocally related PDE systems for a given PDE system has been generalised in series of recent publications of Bluman, Cheviakov and Ivanova [32–35].

The aim of present work is further generalization and development of the methods based on nonlocal transformations of variables for study and integration of nonlinear differential equations. In the current paper we use the classical group-theoretical method of Lie [1–3] and the method of nonlocal transformations of variables [10, 11, 23, 25, 26].

The paper is organized as follows.

In the next section we begin with some preliminary remarks to the method of nonlocal transformations and propose its generalizations. We consider an application of prolonged nonlocal transformations to the prolonged PDEs that do not admit the direct correspondence under appropriate nonlocal transformations in one step but admit such procedure in several steps using the auxiliary intermediate equations.

An existence of the operator equation connecting a *source* equation with a *target* one has allowed us to offer in [36] a method of deriving a solution *adjoint* to solutions of the initial equation. In the present paper we solve this problem differently.

In Section 3 the concept of the nonlocal transformations with additional variables is introduced. The basic formulae of prolongations of them are derived using this notion and different modes of connection of the partial differential equations via such transformations are considered.

Having introduced the concepts above, we establish new algorithms and derive the formulae of generation of solutions to nonlinear equations using transformations with additional variables (Section 4).

An inversion of the nonlocal transformation with additional variables is the subject of Section 5.

2. Some generalizations to nonlocal transformation approach

2.1. Basic concepts of a method of nonlocal transformations

The systematical using of the nonlocal transformations [10, 23, 25, 26, 38] has shown that wide family of known soluble equations admit an adequate interpretation within the framework of a given approach.

Our main goal in this section is to recall the main concepts of the method of nonlocal transformations. Suppose a given nonlocal transformation

$$\mathcal{T}: \quad \begin{array}{ll} x^i = h^i(y, v_{(k)}), & u^K = H^K(y, v_{(k)}), \\ i = 1, \ldots, n, & K = 1, \ldots, m. \end{array} \tag{2.1}$$

maps an *initial* equation

$$F_0(x, u_{(n)}) = 0 \tag{2.2}$$

into an equation $\Phi(y, v_{(q)}) = 0$ of order $q = n + k$, which admits a factorization to another, we call them a *target*, equation

$$F_1(y, v_{(s)}) = 0, \tag{2.3}$$

i.e.,

$$\Phi(y, v_{(q)}) = \lambda F_1(y, v_{(s)}). \tag{2.4}$$

λ is a differential operator of order $n + k - s$. Then we shall say that equations $F_0(x, u_{(n)}) = 0$ and $F_1(y, v_{(s)}) = 0$ are connected by the nonlocal transformation $\mathcal{T}$. Thus, obviously, the identity

$$\mathcal{T} F_0(x, u_{(n)}) \Big|_{DF_1} \equiv 0.$$

is fulfilled. Here and in what follows the symbol $u_{(r)}$ denotes the tuple of partial derivatives of the function u from order zero up to order r. In the case of two independent variables we use the special notation of the variables: $x_1 = x$, $x_2 = t$ and thus $u_t = \partial u/\partial t = \partial_t u$, $u_x = \partial u/\partial x = \partial_x u$.

If the target equation (2.3) coincides with the sours equation (2.2) then $\mathcal{T}$ is a nonlocal invariance transformation of equation (2.2) and we can directly use $\mathcal{T}$ for the construction of a formula generating solutions to this equation.

When (2.3) is linear then we can construct the formulae of nonlinear nonlocal superposition of solutions of equation (2.2). Such formulae allow finding a new solution of equation (2.2) from two known ones and may be thought of as an auto-Bäcklund transformations (ABT) with additional nonlocal variable. Notice also that a nonlocal transformation can map a given sours equation to a target equation which admits additional Lie symmetries. This connection can be used for construction of nonlocal symmetries of the initial equation.

The inversion of a nonlocal transformation is not trivial. Consider two possible approaches solving this problem.

First is based on the integration of the nonlocal substitution as a PDE with respect to a conjugate dependent variable. Let's explain told on a simple example. Given the Cole-Hopf substitution (1.6). To search of a nonlocal substitution in the form $v = H(x, u_{(\pm k)})$, one can integrate (1.6) as differential equation for unknown v with respect to x finding

$$v = \mathrm{e}^{-1/2 \int u(x,t)dx}.$$

Substituting this result into the linear heat equation (1.8) we get an integro-differential expression. Differentiating it with respect to x and simplifying the result we obtain the Burgers equation (1.7).

Often it appears technically impossible to integrate a given substitution with respect to a conjugate dependent variable. Then taking into account that a BT connects given equations in both directions one can try to construct the corresponding BT emanating only from known nonlocal substitution.

So, this approach provides the algebraic resolving of the nonlocal substitution with respect to a derivative of a conjugate dependent variable. Then substitution of the result obtained and its differential consequences into the target equation and solving this for another derivative of the same dependent variable. In such a way one can find the conjugate equation restoring the Bäcklund transformation. The verification of the integrability condition by cross differentiation of obtained expressions must generate the manifold defined by the sours equation.

Assume we have a nonlocal substitution which cannot be solved with respect to the desirable derivative and we aim to receive the BT connecting the given equations. In this case the use of differential consequences of the initial substitution is necessary. For instance, let's derive the BT determined by a system of differential equations of the form (1.2), that is

$$u_x = H^x(x, v_{(k)}), \quad u_t = H^t(x, v_{(k)}),$$

which connects the Burgers equation and the linear heat equation (1.8). Differentiating (1.6) with respect to x and taking into account (1.6) and (1.8) we find

$$u_x = -2\,v^{-2}\left(v\,v_{xx} - 4^{-1}u^2v^2\right).$$

Substituting above expression into the right side of (1.7) and taking into account (1.6), (1.8) again, we get

$$u_t = -v^{-1}\left(2\,v\,v_{xt} + uv_t\right).$$

A cross differentiation of obtained expressions and exclusion of u_x, u_t after simplification, with taking into account of substitution (1.6) in the form $v_x = -2^{-1}uv$, allows to find an expression $\lambda(v_t - v_{xx})$.

2.2. Connection of PDEs by prolonged nonlocal transformations

The efficiency of application the nonlocal transformations for study of nonlinear PDEs were noted above. Nevertheless the invariance of such important equations as Korteweg-de Vries (KdV) equation, sine-Gordon (SG) equation and of others with respect to appropriate nonlocal transformations has appeared possible only by several steps, i.e., using appropriate intermediate equations, which are connected with each other by own nonlocal transformation. Therefore we are very interested in a solution of the corresponding problem of nonlocal invariance by one step. It has appeared that direct nonlocal invariance of such equations or, in some cases, of their differential consequences, becomes possible via appropriate generalization of the approach used.

The SG equation, which first arose in connection with a transformation problem in differential geometry, has long been known to admit a Bäcklund transformations (BT) from which many of its interesting properties were derived.

Example 2.1. Let choose the SG equation in the form

$$u_{xt} - \sin\, u = 0. \tag{2.5}$$

Assume we have only one equation from a system of well known ABT of this equation, and choose it in the form, solved with respect to u_x

$$u_x = -v_x + \frac{2}{b}\,\sin\left(\frac{u-v}{2}\right) = 0. \tag{2.6}$$

Here $v(x,t)$ is a solution of another example of the same equation

$$v_{xt} - \sin\, v = 0. \tag{2.7}$$

Let differentiate both the sides of equality (2.6) with respect to x and substitute the result into (2.5). Then we simplify an obtained result using an equality $v_{xt} = \sin\, v$ and its differential consequences. Solving the above result with respect to u_t we once again differentiate it with respect to x. So we have

$$u_{xt} - \sin\, u = H(u, u_x, v, v_x, v_t, v_{xt}) - \sin\, u. \tag{2.8}$$

Here H denotes for simplicity a quite determined function. Simplifying the right side of (2.8) with use of equations (2.7) and (2.6) we find that a result vanishes identically.

Example 2.2. Consider now the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{2.9}$$

and show an existence of a nonlocal transformation

$$u_x = -(u - v)\sqrt{-2(u+v)} - v_x, \tag{2.10}$$

mapping a given equation into itself

$$v_t + 6vv_x + v_{xxx} = 0. \tag{2.11}$$

To exclude u, u_t from Eq. (2.9) we'll solve it for u and then differentiate a result with respect to x

$$u_x + \frac{1}{6}\frac{u_{xt} + u_{xxxx}}{u_x} - \frac{1}{6}u_{xx}\frac{u_t + u_{xxx}}{{u_x}^2} = 0. \tag{2.12}$$

Substituting $u_t = -6uu_x - u_{xxx}$ into (2.12) we get

$$u_x + \frac{1}{6}\frac{u_{xt} + u_{xxxx}}{u_x} + \frac{uu_{xx}}{u_x} = 0. \tag{2.13}$$

Now we can apply the nonlocal transformation (2.10) to the equation (2.13). Making transition in this result onto the manifold determined by the equation (2.11) and its differential prolongations, and using then equality (2.10) with its differential consequences we get a result, which after simplification vanishes identically.

We could not integrate the equation (2.10) for u and to find in such a way necessary nonlocal substitution. In this case the usage of (2.10) for connection of differential equations can be understood as an application of the prolonged unknown nonlocal transformation to the prolonged PDE.

The examples we had considered allow offering use of prolonged nonlocal transformations in the form

$$\mathcal{T}: \quad x^i = h^i(y, v_{(k)}; u_{(p)}), \quad {u_q}^K = {H_q}^K(y, v_{(k)}; u_{(p)}).$$

Here ${u_q}^K$ are forming an incomplete set of first derivatives in E^m, $i = 1, \dots, n$, $K = 1, \dots, m$, $p \le q < n$ and all possible the integrability conditions must be adopted into consideration.

Thus, we can generalize the factorization scheme in such a way:

$$\mathcal{T}\lambda\ F(x, u_{(n)}) = \lambda^0 F(x, u_{(n)}) + \lambda^1 F_1(y, v_{(s)}) + \lambda^2\mathcal{T}. \tag{2.14}$$

Here λ is the (matrix) operator realizing a necessary prolongation of a given equation (system), $\lambda^2\mathcal{T}$ means $\lambda_K^{2,\,q}(u_q{}^K - H_q{}^K(y, v_{(k)}; u_{(p)}))$. Now the identity

$$\mathcal{T}\lambda F_0(x, u_{(n)})\Big|_{DF_0,\, DF_1,\, D\mathcal{T}} \equiv 0$$

holds. A notation $D(F)$ marks a set defined by a given equation F and by its differential prolongations $D_{(1)}(F)$, $D_{(2)}(F)$ ....

An approach we have considered is a partial realization of the second type of the nonlocal transformations of nonlinear differential equations [25].

2.3. Solution of an initial equation adjoint to its known solutions

This chapter is devoted to the construction of solutions to an initial equation generated by ones of appropriate inhomogeneous target equation. An existence of a factorization equation (2.4) give rise to technique [36] of a finding of a special solution to the initial equation (2.2). Further we name it *adjoint*. The main idea consists of using a nonlocal substitution with a new variable $w(y)$

$$F_1(y, v_{(s)}) = w(y) \tag{2.15}$$

and then a solution of this inhomogeneous PDE with arbitrary perturbation $w(y)$ for v. The problem has appeared rather complicated.

Nevertheless we assume now that a nonlocal substitution $v = V(y, w_{(\pm r)})$ is found and can be applied to equation

$$\lambda(y, v_{(s)})F_1(y, v_{(s)}) = \lambda(y, v_{(s)})\; w(y) = 0.$$

Simplifying the last, one gets an equation for a new variable w

$$\lambda(y, w_{(\pm r+n+k)}(y))\; w(y) = 0.$$

Let's solve an equation obtained and substitute its solution $w = w(y)$ into the equation (2.15). As soon as the solution $v = v(y)$ to this equation may be found, a nonlocal transformation $\mathcal{T}$ allows one to find a solution $u(x)$ of the initial equation (2.2).

Choosing in (2.15) $w(y) = 0$ we get a homogeneous equation (2.3) and, obviously, we have an ordinary case of the conformity of solutions of two given equations (2.2), (2.3). Therefore, a set of solutions of equation (2.2), obtained from the initial solutions of an inhomogeneous equation (2.15), in this sense adjoins to the solutions, which are generated by means of $\mathcal{T}$ from the initial solutions of a homogeneous equation (2.3). How a difficulty of solving Eq. (2.15) can be overcome?

We can solve an above problem by another way. Let's assume, that a given function $v = f(y)$ *is not a solution* of equation (2.3), that is, substituting this function into (2.3), we get the equation (2.15). Suppose, nevertheless that equation (2.4) holds and PDE

$$\lambda(y, v_{(s)})w(y) = 0. \tag{2.16}$$

appears true. Here w runs through the solution set of a *linear* equation with arbitrary variable coefficients. Solving this equation with respect to unknown function w, one can find its solution as a function depending on $v(y)$

$$w = W(y, v(y)). \tag{2.17}$$

After substitution of this expression $w = W(y, v(y))$ into equation (2.15) we obtain an inhomogeneous equation for dependent variable v in the form

$$F_1(y, v_{(s)}) = W(y, v(y)). \tag{2.18}$$

Obviously, that determined by this equation a function $v(y)$ is a solution of equation (2.4). Substituting obtained $v(y)$ into the formulae of a nonlocal transformation $\mathcal{T}$ one can find appropriate solution of a given equation (2.2). Moreover, having the information on point symmetries of the inhomogeneous equation (2.18), one can construct r-parametrical family of solutions to it and, consequently, find the corresponding parametrical sets of solutions to Eq. (2.2). Now we illustrate the construction by some examples.

Example 2.3. It is well known that the Burgers equation (1.7) owing to the Cole-Hopf substitution (1.6) ($v(x,t) \neq 0$) is directly transformable to the linear heat equation [22] as follows

$$\left[-2v^{-1}\partial_x + 2v^{-2}v_x\right](v_t - v_{xx}) = 0. \tag{2.19}$$

Assume there exists a function $w(x,t)$ such that

$$v_t - v_{xx} = w(x,t), \tag{2.20}$$

and solve a linear equation

$$\partial_x w(x,t) - v^{-1}v_x\, w(x,t) = 0. \tag{2.21}$$

with respect to w, having in mind that $v(x,t)$ are variable coefficients. The general solution of this equation can be easily found

$$w(x,t) = s(t)v(x,t).$$

Here $s(t)$ is an arbitrary function. To solve an inhomogeneous equation

$$v_t - v_{xx} = s(t)v(x,t) \tag{2.22}$$

we can choose, for example, one of the group-invariant solutions of equation (2.22). Let it be determined by a characteristic equation

$$\frac{1}{2}\, x\, \partial_x v + t\partial_t v - ts(t)v = 0. \tag{2.23}$$

The solution of a system of differential equations (2.22), (2.23) has the form

$$s(t) = F(t)^{-1} F_t(t), \quad v = F(t)\left(1 + c\ \mathrm{erf}\left(\frac{x}{2\sqrt{t}}\right)\right).$$

Solving the first ordinary differential equation we find $F = c_1 \mathrm{e}^{\int s(t)dt}$, and a function v then reads

$$v = c_1 \mathrm{e}^{\int s(t)dt}\left(1 + c\ \mathrm{erf}\left(\frac{x}{2\sqrt{t}}\right)\right). \tag{2.24}$$

Substituting this result into the Cole-Hopf formula (1.6) we get a corresponding solution of the Burgers equation (1.7)

$$u = -\frac{2\ c_1\ \mathrm{e}^{-\frac{x^2}{4t}}}{\sqrt{\pi t}\left(1 + c_1\ \mathrm{erf}\left(\frac{x}{2\sqrt{t}}\right)\right)}.$$

Let us set in (2.23) $s(t) = \frac{1}{\sqrt{t}}$ and find its solution $v = F(\frac{t}{x^2})\,\mathrm{e}^{2\sqrt{t}}$. Substituting the ansatz obtained into equation

$$v_t - v_{xx} = \frac{1}{\sqrt{t}}\, v(x,t) \tag{2.25}$$

we get the reduction

$$4\omega^2 \ddot{F} + F(6\omega - 1) = 0, \quad \omega = \frac{t}{x^2}.$$

So

$$v = \left(c_1 + c_2\,\mathrm{erf}\left(\frac{1}{2\sqrt{\omega}}\right)\right)\mathrm{e}^{2\sqrt{t}}$$

and consequently

$$u = -\frac{2\ c_2\ \mathrm{e}^{-\frac{x^2}{4t}}}{\sqrt{\pi t}\left(c_1 + c_2\ \mathrm{erf}\left(\frac{x}{2\sqrt{t}}\right)\right)}. \tag{2.26}$$

It was noted above, having information on symmetries of an inhomogeneous equation (2.25), we can generate its parametrical sets of solutions and, consequently, find appropriate parametrical sets of solutions to Eq. (1.7). For example, Lie symmetry generator for (2.25)

$$Y = xt\partial_x + t^2\partial_t - \frac{1}{4}(-4t^{3/2} + x^2 + 2t)v\partial_v$$

allows getting such invariant solution of (2.25):

$$v = -\sqrt{\frac{2}{2-\varepsilon t}}\,\mathrm{e}^{\frac{-16\sqrt{t}+8t^{3/2}\varepsilon - x^2\varepsilon}{4(-2+\varepsilon t)}}\left(-c_1 + c_2\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2\sqrt{t(2-\varepsilon t)}}\right)\right).$$

Here ε is a group parameter. This solution generates appropriate solution to Eq. (1.7)

$$u = \frac{\sqrt{2}}{2}\frac{G}{H},$$

$$G = x\varepsilon\sqrt{2\,\pi t}\left(-c_1 + c_2 \mathrm{erf}\left(\frac{x\sqrt{2}}{2\sqrt{t(2-\varepsilon t)}}\right)\right) + 4c_2\sqrt{2-\varepsilon t}\,\mathrm{e}^{\frac{x^2}{2t(-2+\varepsilon t)}},$$

$$H = \sqrt{\pi t}\,(-2+\varepsilon t)\left(-c_1 + c_2\,\mathrm{erf}\left(\frac{x\sqrt{2}}{2\sqrt{t(2-\varepsilon t)}}\right)\right).$$

One can easy verify this solution cannot be obtained from (2.26) by Lie parametrical generation using the local point symmetry admitted by the Burgers equation.

Another Lie symmetry of equation (2.22) is determined by a characteristic equation

$$t\ x\ \partial_x v + t^2\partial_t v - \frac{1}{4}\left(4t^2 s(t) - 2t - x^2\right) v = 0. \tag{2.27}$$

Choosing here $s(t) = t^{-1}$, we obtain

$$v = (c_1 x + c_2 t)\ t^{-\frac{1}{2}}\ \mathrm{e}^{-\frac{x^2}{4t}},$$

and

$$u = -\frac{c_1(2\ t - x^2) - c_2\ x\ t}{t\ (c_1 x + c_2 t)}.$$

Searching among the non-invariant solutions, we can choose $s(t)$ in the form

$$s(t) = \frac{2 + c_1^2(t + 2\ t^2)}{2\ c_1^2\ t^2}.$$

Appropriate equation (2.22) in this case

$$v_t - v_{xx} = \frac{2 + c_1^2(t + 2\ t^2)}{2\ c_1^2\ t^2}\ v(x,t) \tag{2.28}$$

admits a solution

$$v = c_2\ \mathrm{e}^{-\frac{4t^2 - x^2}{4t}}\ \sin\left(\frac{-x + c_3\ t}{c_1\ t}\right). \tag{2.29}$$

Application of the Cole-Hopf substitution (1.6) to (2.29) allow us to find a corresponding solution of the Burgers equation (1.7)

$$u = \frac{x}{t} + \frac{2}{c_1 t}\cot\left(\frac{-x + c_3\ t}{c_1\ t}\right). \tag{2.30}$$

Example 2.4. Besides linearization by the Cole-Hopf substitution the Burgers equation admits the mapping into itself

$$v_t + v v_x - v_{xx} = 0.$$

via the ABT. Using one equation of this ABT, say

$$u = -2\ \partial_x \ln |v| + v, \tag{2.31}$$

we obtain in this occasion an appropriate operator equation (2.4) [22]

$$[-2v^{-1}\partial_x + 2v^{-2}v_x + 1]\,(v_t + vv_x - v_{xx}) = 0. \tag{2.32}$$

Let us determine the function $w(x,t)$ being a perturbation in the equation

$$v_t + vv_x - v_{xx} = w(x,t) \tag{2.33}$$

and solve a linear equation

$$\partial_x w(x,t) - v^{-1}v_x\ w(x,t) - 2^{-1}w(x,t) = 0 \tag{2.34}$$

with respect to w. A general solution of this equation has a form

$$w = s(t)v\ \exp\left(\frac{1}{2}\int v\ dx\right).$$

Hence the auxiliary function v can be found as a solution of the equation

$$v_t + vv_x - v_{xx} = s(t)\ v\ \mathrm{e}^{\left(\frac{1}{2}\int v\ dx\right)}. \tag{2.35}$$

Solving (2.35) with respect to integral term and differentiating equality with respect to x we get a third order differential equation

$$v_{xt} - \frac{1}{2\ v}\left(v^2v_t + v^3v_x - 3vv_{xx} + 2vv_{xxx} + 2v_tv_x - 2v_xv_{xx}\right) = 0. \tag{2.36}$$

The maximal Lie invariance algebra of (2.36) is spanned by three operators

$$X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = x\partial_x + 2t\partial_t + v\partial_v. \tag{2.37}$$

The characteristic equation associated with X_3

$$x\ v_x + 2\ t\ v_t + v = 0$$

admits a general solution of the form $v = x^{-1}f(t\ x^{-2})$. Substitution of this ansatz into (2.36) yields the reduced ODE depending on argument $w = t\ x^{-2}$

$$-16w^3ff''' + 4w\left((1-44w)f - 3wf^2 + 4w^2f'\right) + 4wf'^2(-1+10w)$$

$$+f'\left(4(1-20w)f + (1-30w)f^2 - 2wf^3\right) - f^4 - 6f^3 - 8f^2 = 0.$$

Solving this equation we find a solution of (2.36)

$$v = -16x^{-1}\frac{-c_1c_3M_1 - U_1 + c_3(\frac{1}{8}+c_1)M_2 - \frac{1}{4}U_2}{c_2\sqrt{tx^{-2}}\ \mathrm{e}^{\frac{x^2}{4t}} + c_3M_1 + U_1}. \tag{2.38}$$

Following notations are used here [37]:

$$M_1 = \mathrm{KummerM}(1-4c_1, 3/2, x^2/4t), \quad M_2 = \mathrm{KummerM}(-4c_1, 3/2, x^2/4t),$$

$$U_1 = \mathrm{KummerU}(1-4c_1, 3/2, x^2/4t), \quad U_2 = \mathrm{KummerU}(-4c_1, 3/2, x^2/4t)$$

and c_1, c_2 and c_3 are arbitrary constants.

Note that substitution of the same ansatz $v = x^{-1} f(t\, x^{-2})$ into (2.35) allows us obtain a reduction to the ODE with integral term only for case $s(t) = t^{-1}$. As a result we get a solution of equation (2.35) in the form (2.38) again.

Inserting obtained v into the nonlocal transformation (2.31) we easily find corresponding solution of the Burgers equation

$$u = \frac{\mathrm{e}^{\frac{x^2}{4t}} t G_1 + x^2\sqrt{t/x^2}\, G_2}{xt\sqrt{t/x^2}\,(c_2\sqrt{t/x^2}\,\mathrm{e}^{\frac{x^2}{4t}} + c_3 M_1 + U_1) G_3}. \tag{2.39}$$

Such notations are used here:

$$\begin{aligned} G_1 &= 8c_1c_2c_3M_2 - 2c_2U_2 + c_2c_3M_2, \\ G_2 &= c_3^2 M_1 M_2 + c_3 U_1 M_2 - 2c_3 U_2 M_1 \\ &\quad +8c_1c_3^2 M_1 M_2 + 8c_1c_3U_1M_2 - 2U_1U_2, \\ G_3 &= -8c_1c_3M_1 - 8c_1U_1 + c_3(1+8c_1)M_2 - 2U_2. \end{aligned}$$

As it has been shown above, when the nonlocal transformation $\mathcal{T}$ connects two given equations as follows

$$F_0(x, u_{(n)}) = \lambda F_1(y, v_{(s)}), \tag{2.40}$$

there exists a technique, which allows to construct another solution of a given equation F_0 from any given solution of the *inhomogeneous* equation $F_1 = w$ with *special* right side.

Note. Here we aim to point out how an existence of the nonconserved currents or nonvanishing curvature connected with the target equation give rise to adjoint solution of the initial equation. We are mainly interested in the role which the conservation laws play within a nonlocal transformation approach.

We shall deal first with the case when the partial differential equation in the simplest case of two independent variables

$$F(x, t, u_{(k)}) = 0 \tag{2.41}$$

should admit at least one conservation law

$$D_t \phi^t(x, t, u_{(r)}) + D_x \phi^x(x, t, u_{(r)}) = 0. \tag{2.42}$$

Here D_t and D_x are the total derivatives with respect to the variables t and x, ϕ^t and ϕ^x are conserved density and flux, respectively. Conservation laws play an important role in the understanding of the physical model associated with a system of PDE and can be useful for the symmetry searching and effective integration of such equations.

In order to present Eq. (2.41) in the conservation law form a potential function v determined by the auxiliary system

$$v_x = \phi^t(x, t, u_{(r)}), \quad v_t = -\phi^x(x, t, u_{(r)}). \tag{2.43}$$

should be introduced [8]. It is worth mentioning that the notion of potential symmetries of differential equations was introduced by Bluman *et al.* [3, 8]. The study of system (2.43) within the framework of the classical group analysis as a rule result in an additional information about symmetry properties of the initial equation (2.41). The symmetry is said to be a *potential symmetry* of (2.41) when it is a point symmetry of a potential system (2.43) which is not projected onto a point symmetry of (2.41).

Excluding the initial dependent variable u from the system (2.43) it is often possible to obtain an equation

$$G(x, t, v_{(k)}) = 0 \tag{2.44}$$

with respect to only the potential variable v. Then this system represents a Bäcklund transformation connecting (2.41) with (2.44). This is the case we are interested in.

Assume now that given equations (2.2) and (2.3) are connected by a nonlocal transformation $\mathcal{T}$ and each of them admits appropriate conserved current. Then operator equation

$$F_0(x, u_{(n)}) = \lambda\, F_1(y, v_{(s)}), \tag{2.45}$$

gives rise to the corresponding operator expression for conserved currents

$$\mathrm{div} J_{(u)}(x, u_{(n)}) = \lambda(y, v_{(s)})\, \mathrm{div} J_{(v)}(y, v_{(s)}). \tag{2.46}$$

In two-dimensional case this means

$$D_t f^t(x, t, u_{(r)}) + D_x f^x(x, t, u_{(r)})$$
$$= \lambda(y, \tilde{t}, v_{(s)}) \left(D_{\tilde{t}} g^{\tilde{t}}(y, \tilde{t}, v_{(k)}) + D_y g^y(y, \tilde{t}, v_{(k)}) \right),$$

where f^t, f^x and $g^{\tilde{t}}, g^y$ are conserved density and flux for each equation respectively. This gives rise to nonlocal correspondence between the potential functions that can be obtained for each conservation law, just as it has been made for the system (2.43).

As we have shown above, an adjoint solution of the equation (2.2) follows from the inequality

$$J_{(v)}(y, v_{(s)}) \neq 0$$

with the special choice of the right term in the target equation. Hence, the least action principle for the equation (2.3) admits appropriate deviations, determined by adding into the model of external charge. This turns Eq. (2.3) into an inhomogeneous one. In other words, an adjoint solution of the equation F_0 can be generated by a solution of the perturbed equation F_1, last may be thought of as F_1 putted into the special exterior field, generated by an appropriate density of a charge.

On the other hand the theory of the vector bundle in differential geometry is the mathematical basis of gauge field theory, where a connection on a vector bundle may be thought of as a gauge potential, and the curvature as a field strength. Hence application of the appropriate geometrical model to equation $divJ_{(v)}(y, v_{(s)}) = 0$ allows interpret it as the zero curvature condition for suitable connections [21]. We conclude, therefore, that the aforementioned solutions arise from the *nonzero curvature condition* based on an inhomogeneous equation with special form of the right hand side. By virtue of noted above conserved currents (derived from symmetries of the action principle) and the vector bundle approach with theory of connections became an effective tool for the actual integration of the differential equations.

3. Nonlocal transformations with additional variables

We aim here to develop a technique of introduction additional variables into a nonlocal transformation. Before making any definitions we first introduce some notation. We'll write a given system of PDE, using the subscripts to denote the partial derivatives

$$\begin{aligned} &{F_0}^A(x, u_{(k)}) = 0, \quad x = \{x^i\}, \quad i = 1 \dots n, \\ &A = 1 \dots M, \quad u = \{u^K(x)\}, \quad K = 1, \dots, N. \end{aligned} \tag{3.1}$$

For simplicity we will use also a designation

$$\{{F_0}^A(x, u_{(k)})\} = F_0(x, u_{(k)}).$$

Let's assume a nonlocal transformation is given

$$\begin{aligned} \mathcal{T}: \quad x^i = h^i(y, v_{(k)}; z^{(G)}, {w_{(q)}}^{(G)}), \quad u^K = H^K(y, v_{(k)}; z^{(G)}, {w_{(q)}}^{(G)}), \\ z^{(G)\ j} = \varphi^j(y, v_{(k)}; {w_{(q)}}^{(G)}), \quad G = 1, \dots, m. \end{aligned} \tag{3.2}$$

Here, unlike a nonlocal transformation $\mathcal{T}$ (2.1), the functions on the right of equations depend on new variables ${w_{(q)}}^{(G)}(z^{(G)})$ which depend on own new independent variables $z^{(G)}$, external with respect to (y, v). Besides, new expressions defining new independent variables are present in (2.1).

A differential prolongation of the finite nonlocal transformation $\mathcal{T}$ gives rise getting appropriate expressions for the first derivatives, and then, applying a prolongation procedure repeatedly, one can get appropriate expressions for derivatives of the second order and so on. For example, to find the first derivatives we consider n equations obtained from (3.2) by differentiation with respect to x^j

$$D_i H^K(y, v_{(k)}; z^{(G)}, w_{(q)}{}^{(G)}) = u_j D_i h^j(y, v_{(k)}; z^{(G)}, w_{(q)}{}^{(G)}). \tag{3.3}$$

If we denote $\widetilde{D}_i$ and $\widetilde{D}_G$ the total derivatives with respect to variables y and z accordingly

$$\widetilde{D}_i = \partial_i + \partial_i v_{(p)}\ \partial_{v_{(p)}} + \ldots, \quad \widetilde{D}_Q = \partial_{z^{(Q)}} + \partial_{z^{(Q)}} w_{(q)}{}^{(T)}\ \partial_{w_{(q)}{}^{(T)}} + \ldots$$

and use the summation convention for repeated indices, the left side of equation (3.3) can be written in a form

$$D_i H^K = \widetilde{D}_i H^K + D_i z^{(G)} \widetilde{D}_Q H^K.$$

Solving a system of linear algebraic equations (3.3) with respect to u_j we get the transformation formulae for all first derivatives

$$u_j = H_j{}^K(y, v_{(k+1)}; z^{(G)}, w_{(q+1)}{}^{(G)}),$$

i.e. the first prolongation of (3.2) is obtained. When applied recursively in this case, such successive prolongation adds new derivatives to the previous ones, and so generates, after r applications, the $r+1$ -derivatives

$$u_{(r)\ j} = H^K{}_{(r)\ j}(y, v_{(k+r+1)}; z^{(G)}, w_{(q+r+1)}{}^{(G)}).$$

So, it is obvious that a nonlocal transformation (3.2) if being applied to a given equation (3.1) maps it into the higher order equation

$$\Omega(y, v_{(n+k)}; z^{(G)}, w_{(q+k)}{}^{(G)}) = 0,$$

which may admit a factorization to a tuple of equations of different types:

$$F_1(y, v_{(s)}) = 0, \tag{3.4}$$

$$F_J(z^{(J)}, w_{(q+k)}{}^{(J)}) = 0, \quad J = 2, \ldots, m, \tag{3.5}$$

(according to our notation, (3.5) should be understood as a system of equations $F^B{}_J(z^{(J)}, w_{(q+k)}{}^{(J)}) = 0$), and, at last, a set of equations of the more general form

$$\Phi_C(y, v_{(n+k)}; z^{(G)}, w_{(q+k)}{}^{(G)}) = 0, \quad C = m, \ldots, c. \tag{3.6}$$

Here we suppose all $w_{(q+k)}{}^{(G)}$ depend on full set of independent variables $y,\ z^{(G)}$.

In other words, we suppose that factorization admits a presentation by the operator expression

$$\begin{aligned} &\Omega(y, v_{(n+k)}; z^{(G)}, w_{(q+k)}{}^{(G)}) = \lambda^1 F_1(y, v_{(s)}) + \\ &\lambda^J F_J(z^{(J)}, w_{(q+k)}{}^{(J)}) + \lambda^C \Phi_C(y, v_{(n+k)}; z^{(G)}, w_{(q+k)}{}^{(G)}). \end{aligned} \tag{3.7}$$

Here the summation convention for repeated indices has been used,

$$\lambda^Q(y, v_{(n+k)}; z^{(G)}, {w_{(q+k)}}^{(G)})$$

are the differential operators of appropriate order. Then we say that equations $F_0(x, u_{(n)}) = 0$ and $F_1(y, v_{(s)}) = 0$ *are forced* to be connected by the nonlocal transformation $\mathcal{T}$, which depends on additional variables, and the equation

$$\mathcal{T} F_0(x, u_{(n)}) \Big|_{DF_1, DF_J, D\Phi_C} \equiv 0$$

is fair. As well as earlier, $D(F)$ means a set of a given equation F and its necessary differential prolongations. This identity means a transition in the result of application of the nonlocal transformation to Eq (3.1) onto the manifold defined by equations (3.4), (3.5) and (3.6) and their necessary differential prolongations. The simplified result vanishes identically. Without the second and the third terms in the right hand side in (3.7) the nonlocal connection between equations F_0 and F_1 we assume impossible. So, they force this equations to be connected and we can call this case "the forced nonlocal connection".

Setting $\lambda^J = \lambda^C = 0$ in (3.7) we obtain an ordinary nonlocal transformation $\mathcal{T}$ described at the beginning of Section 2.

In that case, when $\lambda^C = 0$, a given equation (3.1) admits a mapping into the equation (3.4) by means of expansion of the nonlocal transformation by a set of new dependent variables ${w_{(q)}}^{(G)}(z^{(G)})$, each of which depends on appropriate set of own additional independent variables $z^{(G)}$. Each dependent variable is determined by own PDE of the form (3.5).

Suppose that $\lambda^J = 0$. Then an equation (3.1) can be connected with an equation (3.4) by the nonlocal transformation, the supplemented by a set of new dependent variables ${w_{(q)}}^{(G)}(z^{(G)})$, that depend on full set of all independent variables y, $z^{(G)}$. If it is possible to hunt out the solution of a system (3.4), (3.6), the corresponding solution of a given equation (3.1) can be found. In particular, system (3.6) can appear linear, where variable factors are some functions of y and $v_{(q)}$.

Notice that a concept of an adjoint solution which we had developed in the second subsection of Section 2. can be applied to the generalized nonlocal transformation (3.7). Set, for example,

$$\begin{gathered} F_1(y, v_{(s)}) = W_1(y), \quad F_J(z^{(J)}, {w_{(q+k)}}^{(J)}) = W_J(z^{(J)}), \\ \Phi_C(y, v_{(n+k)}; z^{(G)}, {w_{(q+k)}}^{(G)}) = W_C(y, z^{(G)}), \end{gathered} \tag{3.8}$$

were $W_1(y)$, $W_J(z^{(J)})$ and $W_C(y, z^{(G)})$ are arbitrary perturbations of the corresponding equations. Then a condition $\Omega(y, v_{(n+k)}; z^{(G)}, {w_{(q+k)}}^{(G)}) = 0$ leads to the equation

$$\lambda^1 W_1(y) + \lambda^J W_J(z^{(J)}) + \lambda^C W_C(y, z^{(G)}) = 0. \tag{3.9}$$

If this equation can be solved with respect to $W_{\dots}$, then the possibility to construct an adjoint solution of equation (3.1) arises.

Let's consider now examples of the forced connection of the nonlinear partial differential equations via the nonlocal transformation with additional variables.

Example 3.1. Suppose the Burgers equation (1.7) admits a linearization by a nonlocal transformation with one additional variable to the linear heat equation (1.8) jointly with auxiliary linear equation depending on additional function $z(x,t)$

$$z_t - {v_x}^2 - z_{xx} = 0. \tag{3.10}$$

Let's search for this nonlocal transformation in the form

$$u(x,t) = H(v(x,t), v_x(x,t), z(x,t), z_x(x,t)). \tag{3.11}$$

Substituting (3.11) into the equation (1.7) we obtain an expression, which depends on the third order derivatives. Making transition here onto the manifold defined by equations (1.8), (3.10) and by their differential prolongations with respect to x, and simplifying the result, we get a differential expression. This expression determines a function $H(v,p,z,q)$, $p = v_x$, $q = z_x$, and splitting it with respect to the derivatives

$$v_{xx}(x,t),\ v_{xxx}(x,t),\ z_{xx}(x,t),\ z_{xxx}(x,t),$$

we get the following system of partial differential equations:

$$\begin{aligned} &p\,HH_v + (qH + p^2)H_z - 2\,p\,qH_{vz} - q^2 H_{zz} - p^2 H_{vv} = 0,\\ &-2\,q\,H_{zp} + HH_p + 2\,p\,H_q - 2\,p\,H_{vp} = 0,\\ &-2\,p\,H_{vq} + HH_q - 2\,q\,H_{zq} = 0, \quad H_{pp} = H_{qq} = H_{pq} = 0. \end{aligned} \tag{3.12}$$

The solutions of the system (3.12) can be easily found. The first solution when $H_z = 0$ is the Cole-Hopf substitution

$$u = -\frac{2\,v_x}{v - 2\,c}. \tag{3.13}$$

Another solution has a form

$$u = -\frac{(-4\,v + 8\,c_1)\,v_x - 4\,z_x}{2\,z + v^2 - 4\,c_1 v - 4\,c_2}. \tag{3.14}$$

An existence of this nonlocal connection may be used for construction new technique of generation of solutions to the Burgers equation (1.7).

Example 3.2. It is well known (see e.g. [12]) that nonlinear heat equation

$$u_t - \partial_x(u^{-2}u_x) = 0 \tag{3.15}$$

admits *linearization* by a nonlocal transformation. Let $u(x,t)$ is solution of an initial equation (3.15) and $v(y,t)$ is a solution of the target equation

$$v_t - \partial_x(v^{-2}v_x) = 0.$$

One can easily verify a nonlocal *invariance* of the equation (3.15) via the transformation with additional variables

$$u(x,t) = \frac{1}{\frac{1}{v(y,t)} + D_1 w\,(z,t)}, \quad x = y + w\,(z,t)\,, \tag{3.16}$$

where a new additional independent variable is determined by a relation $z = \int v(y,t)dy + f(t)$ and $y = \phi(x,t)$ is defined by the differential equation

$$\phi_t - \frac{\phi_{xx}}{(v(\phi,t)\phi_x)^2} = 0. \tag{3.17}$$

Here the additional variable $w\,(z,t)$ runs through the set of solutions of the linear heat equation $D_2 w - D_{11} w = 0$.

Using a connection of a linear heat equation with the Burgers equation (1.6) one can easy construct another nonlocal invariance transformation of (3.15) with additional variable being a solution of the Burgers equation

$$u(x,t) = \frac{1}{\frac{1}{v(y,t)} - \frac{1}{2} w\,(z,t)\,\mathrm{e}^{-\frac{1}{2}\int^z w(r,t)dr}}, \tag{3.18}$$

$$x = y + \mathrm{e}^{-\frac{1}{2}\int^z w(r,t)dr}, \quad z = \int v(y,t)dy.$$

A new additional independent variable is defined as well as in the previous case by a relation $z = \int v(y,t)dy + f(t)$ and $y = \phi(x,t)$ is determined by the equation (3.17). An additional variable $w\,(z,t)$ runs through the set of solutions of the equation

$$D_2 w + w D_1 w - D_{11} w = 0.$$

As has been pointed above, the nonlocal transformations with additional variables can be effectively applied. In the next section we'll develop a new approach for generation of solutions to given equation from known ones. When using a nonlocal (invariance or linearization) transformation with the additional variables we assume that an initial equation do not admits marked properties without them. Consequently they do forcing a given equation to this behavior (property). The "mechanism" of this forcing lays in the factorization equality which is described by means of an operator expression (3.7) as well as in an interpretation of an adjoint solution in Note to Subsection 2.3. A notions "gauge symmetry", "gauge transformation" are rather close but have quite special application.

4. Forced symmetries and generation of solutions

Variety of generation methods have been developed for solving the extensive classes of nonlinear partial differential equations. They often result in algorithms for finding new solutions from known one. Application of nonlocal transformations with additional variables allows constructing a new technique of generation. As we use a nonlocal mapping of an initial equation onto a given one *under force* determined by additional function (a solution of additional equation), the appropriate nonlocal symmetries we call the *forced symmetries*.

In this section we shall demonstrate the efficiency of formulae obtained above for construction of new solutions, depending on an arbitrary function, for a given equation.

We show first how to derive (explicit) solutions of Eq. (1.7) directly from the formula (3.14) using an arbitrary solution $v(x,t) = \phi(x,t)$ of a linear heat equation. Having such a solution we then substitute its derivative into Eq. (3.10). Solving this equation for $z(x,t)$ and substituting the components obtained into (3.14), we find necessary solution. It is obvious that the basic complexity is related with finding solutions of the equation

$$z_t - \phi_x(x,t)^2 - z_{xx} = 0. \tag{4.1}$$

Here are some examples of construction of solutions to Eq. (1.7).

1. $v = x^2 + 2t, \quad \rightarrow$

$$z = c_4 G_1 \mathrm{e}^{k\,t} - \frac{c_1 + c_2 x^4 + c_3 x}{3c_2} \quad \rightarrow$$

$$u = -4\left(2c_2x^3 + 12c_2x(t-k_1) + 3\sqrt{k}c_2c_4\mathrm{e}^{k\,t}G_2\right)$$
$$\times\left[6c_2c_4\mathrm{e}^{k\,t}G_1 + c_2x^4 + 12c_2(x^2+t)(t-k_1) - 2c_3x - 12k_1c_2t - 12k_2c_2 - 2c_1\right]^{-1}.$$

Here,

$$G_1 = c_5\mathrm{e}^{\sqrt{k}\,x} + c_6\mathrm{e}^{-\sqrt{k}\,x} \quad \text{and} \quad G_2 = c_5\mathrm{e}^{\sqrt{k}\,x} - c_6\mathrm{e}^{-\sqrt{k}\,x},$$

and c_i, k, k_j are some constants.

2. $v = k\sin(\omega)\,\mathrm{e}^{a\,t}, \quad \rightarrow$

$$z = c_4c_6\mathrm{e}^{c_1t-\sqrt{c}_1x} + c_4c_5\mathrm{e}^{c_1t+\sqrt{c}_1x} - \tfrac{k^2}{2}\mathrm{e}^{2at}$$
$$+\tfrac{1}{2c_3}\left(c_1k^2\mathrm{e}^{2at}\cos(\sqrt{2}\,\omega) + c_2k^2\mathrm{e}^{2at}\sin(\sqrt{2}\,\omega) + c_3k^2\mathrm{e}^{2at}\cos^2(\omega)\right).$$

The notation $\omega = \sqrt{-a}\,x + b$ is used here. $\rightarrow \quad u = \frac{A}{B}$,

$$A = 4c_3\left(2kk_1\mathrm{e}^{at}\sqrt{-a}\cos(\omega) + c_4c_6\sqrt{c}_1\mathrm{e}^{c_1t-\sqrt{c_1}\,x} - c_4c_5\sqrt{c_1}\,\mathrm{e}^{c_1t+\sqrt{c_1}\,x}\right)$$
$$+4k^2\,\mathrm{e}^{2at}\sqrt{-a}\left(c_1\sin(\sqrt{2}\,\omega) - c_2\cos(\sqrt{2}\,\omega)\right),$$

$$B = 2c_3c_4c_6\mathrm{e}^{c_1t-\sqrt{c_1}x} + 2c_3c_4c_5\mathrm{e}^{c_1t+\sqrt{c_1}x} - 4kk_1c_3\mathrm{e}^{at}\sin(\omega)$$

$$+k^2\mathrm{e}^{2at}\left(c_2\sin(\sqrt{2}\,\omega) + c_1\cos(\sqrt{2}\,\omega)\right) - 4k_2c_3.$$

3. $v = x, \quad \to \quad z = -\dfrac{c_1}{2}x^2 + t(1-c_1) \quad \to$

$$u = \frac{4\left(x(1-c_1) - 2k_1\right)}{-(x^2+2t)(1-c_1) + 4(k_1x+k_2)}.$$

To develop analytical techniques for generation of solutions to Eq. (1.7) starting from a formula (3.14)

$$u^{\mathrm{II}} = -\frac{(-4\,v + 8\,c_1)\,v_x - 4\,z_x}{2\,z + v^2 - 4\,c_1v - 4\,c_2}, \tag{4.2}$$

we shall build in into it another solution of this equation. If we set

$$v_x = -\frac{1}{2}u^{\mathrm{II}}v$$

in (4.2) the formula for generation of solutions will be read

$$u^{\mathrm{I}} = -\frac{(-2\,v + 4\,c_1)\,u^{\mathrm{II}}\,v - 4\,z_x}{2\,z + v^2 - 4\,c_1v - 4\,c_2}. \tag{4.3}$$

Solving this expression with respect to u^{II}, we get

$$u^{\mathrm{II}} = \frac{1}{2}\frac{(2\,z + v^2 - 4\,c_1v - 4\,c_2)\,u^{\mathrm{I}}\,v + 4\,z_x}{v(v-2\,c_1)}. \tag{4.4}$$

Now it is possible to construct new solutions u^{II} of equation (1.7) from the known ones u^{I} choosing an arbitrary known solution of the linear heat equation $v(x,t)$. The appropriate auxiliary function $z(x,t)$ may be found as a solution of the PDE

$$g(x,t) = -\frac{(-4\,\phi(x,t) + 8\,c_1)\,\phi_x(x,t) - 4\,z_x}{2\,z + \phi(x,t)^2 - 4\,c_1\phi(x,t) - 4\,c_2}, \tag{4.5}$$

where $u^{\mathrm{I}} = g(x,t)$ is the known solution of (1.7) and $v(x,t) = \phi(x,t)$ is an arbitrary solution of the linear heat equation (1.8). Solving this differential equation for $z(x,t)$, we find an expression which contains an arbitrary function of t. This function can be specialized by the equation (3.10). Inserting the components obtained into (4.4), we find the searched.

Theorem 4. 1. *The generating solutions formula for equation (1.7) has the form*

$$u^{\mathrm{II}} = \frac{1}{2}\frac{(2\,-\tau + v^2 - 4\,c_1v - 4\,c_2)\,u^{\mathrm{I}}\,v + 4\,\tau_x}{v(v-2\,c_1)}, \tag{4.6}$$

where $u^{\mathrm{I}} = g(x,t)$ *is the known solution of (1.7) and* $v(x,t) = \phi(x,t)$ *is arbitrary solution of the linear heat equation* $v_t(x,t) - v_{xx}(x,t) = 0$. *The functional parameter* $\tau(x,t)$ *is constructed in the following way: Given a solution* $u^{\mathrm{I}} = g(x,t)$ *of Eq. (1.7), we solve the partial differential equation*

$$g(x,t) = -\frac{(-4\,\phi(x,t) + 8\,c_1)\,\phi_x(x,t) - 4\,\tau_x}{2\,\tau + \phi(x,t)^2 - 4\,c_1\phi(x,t) - 4\,c_2}, \tag{4.7}$$

with respect to the function $\tau(x,t)$ *containing an arbitrary function of* t *which is specialized by the equation*

$$\tau_t - \phi_x(x,t)^2 - \tau_{xx} = 0. \tag{4.8}$$

Here we present several examples on application of a formula obtained. Having in mind reduction of an account the corresponding equations for additional function z and their solutions are lowered in what follows.

1. $u^{\mathrm{I}} = \dfrac{x}{t}, \quad v = x^2 + 2t \quad \to \quad u^{\mathrm{II}} = -\dfrac{2}{x}.$
2. $u^{\mathrm{I}} = \dfrac{x}{t}, \quad v = \dfrac{k}{\sqrt{t}}\,\mathrm{e}^{-\frac{x^2}{4t}} \quad \to \quad u^{\mathrm{II}} = \dfrac{x}{t}.$
3. $u^{\mathrm{I}} = \dfrac{x}{t}, \quad v = 2\sqrt{t}\,\mathrm{e}^{-\frac{x^2}{4t}} + \sqrt{\pi}\,x\,\mathrm{erf}\left(\dfrac{1}{2\sqrt{\frac{t}{x^2}}}\right) \quad \to$

$$u^{\mathrm{II}} = -\frac{2\sqrt{\pi}\,\mathrm{erf}\left(\frac{1}{2\sqrt{\frac{t}{x^2}}}\right)}{2\sqrt{t}\,\mathrm{e}^{-\frac{x^2}{4\,t}} + \sqrt{\pi}\,x\,\mathrm{erf}\left(\frac{1}{2\sqrt{\frac{t}{x^2}}}\right)}.$$

4. $u^{\mathrm{I}} = -\dfrac{2}{x}, \quad v = x \quad \to \quad u^{\mathrm{II}} = -\dfrac{4x}{x^2 + 2t}.$

We have so far dealt with the nonlocal transformations, in which the independent variables remained unchangeable. Now we turn our attention to the formulae (3.16). It is nothing other than the formula for generation of solutions to Eq. (3.15) quite ready to application.

As was noted above, the equation (3.15) admits *linearization* [38] via nonlocal transformation of special sort. In considered above case we have got a nonlocal *invariance* of this equation *under force* of an additional function, which is an arbitrary known solution of the linear heat equation.

Theorem 4. 2. *The formula for generation of solutions to equation (3.15) is*

$$u^{\mathrm{II}}(x,t) = \frac{1}{\frac{1}{u^{\mathrm{I}}(\tau^{\mathrm{I}},t)} + D_1 z\left(\int u^{\mathrm{I}}(\tau^{\mathrm{I}},t)d\tau^{\mathrm{I}} + f(t), t\right)}, \tag{4.9}$$

where the functional parameter τ^{I} *can be obtained solving the equation*

$$\tau^{\mathrm{I}} - x + z\left(\int u^{\mathrm{I}}(\tau^{\mathrm{I}}, t)d\tau^{\mathrm{I}} + f(t), t\right) = 0,$$

an arbitrary function $f(t)$ *is specialized by the equation (3.17)*

$$\tau^{\mathrm{I}}_t - \frac{\tau^{\mathrm{I}}_{xx}}{(u^{\mathrm{I}}(\tau^{\mathrm{I}}, t)\tau^{\mathrm{I}}_x)^2} = 0. \tag{4.10}$$

Additional variable $z\left(\tau^{\mathrm{II}}, t\right)$ *runs through the set of solutions of the linear heat equation* $D_2 z - D_{11} z = 0$.

The formulae derived are effectively used for the construction of exact solutions. Let us give some examples of application of this algorithm.

1. $u^{\mathrm{I}} = 1, \quad z = x^2 + 2t \quad \rightarrow \quad u^{\mathrm{II}} = \dfrac{1}{\sqrt{4x - 8t + 4c + 1}}.$

2. $u^{\mathrm{I}} = -\dfrac{2}{x}, \quad z = x^2 + 2t \quad \rightarrow \quad u^{\mathrm{II}} = -\dfrac{2}{\mathrm{e}^w + 8w - 2t - 4c}$, where the function w is determined by equation

$$4\mathrm{e}^w + 16w^2 - 8wt - 16cw - 4x + 8t + (t + c)^2 = 0.$$

3. $u^{\mathrm{I}} = -\dfrac{1}{x + c}, \quad z = x^2 + 2t \quad \rightarrow \quad u^{\mathrm{II}} = -\dfrac{1}{\mathrm{e}^G + 2G - 2t},$

$$\mathrm{e}^G + (G - t)^2 - x + 2t - c = 0.$$

4. $u^{\mathrm{I}} = h, \quad z = \sqrt{\pi}\mathrm{erf}\left(\dfrac{x}{2\sqrt{t}}\right) \quad \rightarrow \quad u^{\mathrm{II}} = \dfrac{h\sqrt{t}}{\sqrt{t} + h\mathrm{e}^{-\frac{h^2 Q^2}{4t}}},$

$$Q - x + \sqrt{\pi}\mathrm{erf}\left(\frac{hQ + c_1}{2\sqrt{t}}\right) + c_1 = 0.$$

5. $u^{\mathrm{I}} = -\dfrac{2c_1}{c_1 x + c_2}, \quad z = x \quad \rightarrow$

$$u^{\mathrm{II}} = \left(\mathrm{Lambert}\left(-\frac{\exp(\frac{c_1(-2x + t + 2c_3) - 2c_2}{4c_1})}{2c_1}\right) + 1\right)^{-1}.$$

So we can conclude that proposed additional nonlocal symmetry allows the generation into more wide families of exact solutions of nonlinear differential equations. All solutions found can be naturally extended to multi-parameter families of solutions by means of the Lie symmetry transformations or by any other formulae, enabling a generation of solutions. Some of them can be obtained in an explicit form, while the other may have a parametrical representations with functional parameters given in implicit form. Showing examples obtained above, we do not claim to supplement sets of new exact solutions of cited equations. We aim to reveal the efficiency of algorithms proposed and establish their complexity degree.

5. On inversion of the nonlocal transformation with additional variables

We shall devote this section to defining inversion of the nonlocal transformation with additional variables. The problem that arises now is whether or not a given nonlocal transformation with additional variables has an inverse. The inversion of nonlocal transformation with additional variables is not trivial and may be defined in either of two ways (were pointed out above in Section 2).

Let us begin with an approach, based on the integration of the nonlocal substitution as a PDE for one of dependent variables. We understand other dependent variables as variable coefficients. Then substitution of the solution obtained into the equation for this variable enables us to get an operator expression for all the rest equations. The last can be eliminated on prolonged manifolds defined by such equations. Notice, that there exists obvious a technical problem which has to be overcome. It is an integration of appropriate PDE.

The concepts introduced above may be exemplified. Let us first integrate a substitution (3.14)

$$u = -\frac{(-4\,v + 8\,c_1)\,v_x - 4\,z_x}{2\,z + v^2 - 4\,c_1 v - 4\,c_2} \tag{5.1}$$

with respect to v. We find a such solution:

$$v = 2k_1 - \sqrt{4{k_1}^2 + s(t)e^{-1/2\int u(x,t)dx} - 2z(x,t) + 4k_2}. \tag{5.2}$$

Setting here $s(t) = 1$ we aim substitute v into the linear heat equation (1.8), but Eq. (3.10) depends on the same variable v. To overcome this technical problem we shall differentiate an expression (5.2) with respect to x and again use (5.2) for elimination v. So we get

$$v_x = -\frac{1}{2}\left(-\frac{1}{2}ue^{-1/2\int u(x,t)dx} - 2z_x\right) \times \left[\sqrt{4{k_1}^2 + e^{-1/2\int u(x,t)dx} - 2z(x,t) + 4k_2}\right]^{-1}. \tag{5.3}$$

Now the equation (3.10) becomes

$$z_t - \frac{1}{4}\frac{\left(-\frac{1}{2}ue^{-1/2\int u(x,t)dx} - 2z_x\right)^2}{4{k_1}^2 + e^{-1/2\int u(x,t)dx} - 2z(x,t) + 4k_2} - z_{xx} = 0. \tag{5.4}$$

Making a change of variable (5.2) in Eq. (1.8) and substituting into the result obtained z_{xx} from the above equation we find an integral consequence of the third Eq. (1.7)

$$\int u_t(x,t)\,dx + \frac{1}{2}u^2 - u_x = 0.$$

Solving substitution (3.14) with respect to z we find

$$z = \left(\int \left(-vv_x - \frac{1}{4}uv^2 + k_1uv + k_2u + 2k_1v_x\right) W\,dx\right) W^{-1}, \tag{5.5}$$

$$W = \mathrm{e}^{1/2\int u(x,t)dx}.$$

Application (5.5) for Eq. (3.10) and insertion $v_t = v_{xx}$ allow us to obtain an integro-differential expression. Elimination of integral terms in last leads us to the integro-differential consequences of the Burgers equation (1.7).

The alternative approach, which we shall adopt here, is based on the construction of the special BT connecting the given equations. Let's assume, that a given equation (3.14) is a first expression for required BT. Differentiating it with respect to x, then inserting $v_{xx} = v_t$, $z_{xx} = z_t - {v_x}^2$ and u from (3.14), we get a conjugate equation for such a BT

$$\begin{aligned}
&u_x = -\gamma^{-2}\Lambda,\\
&\gamma = 2z + v^2 - 4k_1v - 4k_2,\\
&\Lambda = (-2v^2 + 8k_1v - 8{k_1}^2){v_x}^2 + (8k_1 - 4v)v_xz_x - 2{z_x}^2\\
&+(v^3 + 2vz - 6k_1v^2 + 8{k_1}^2v - 4k_2v - 4k_1z + 8k_1k_2)v_t\\
&+(v^2 + 2z - 4k_1v - 4k_2)z_t.
\end{aligned} \tag{5.6}$$

Substituting u defined by (3.14) into (5.6) obtained above we get an expression vanishing on the manifold defined by equations (1.8) and (3.10).

Now we shall be concerned only with inversion of the BT obtained what may be thought of as construction of another BT allowing to exclude a variable v. With that end in view we solve (5.2) with respect to an exponential term

$$e^{-1/2\int u(x,t)dx} = 2z + v^2 - 4k_1v - 4k_2 \tag{5.7}$$

and substitute it into (5.3)

$$\begin{aligned}
v_x = &-\frac{1}{2}\left(-\frac{1}{2}\,u\,(2\,z + v^2 - 4k_1v - 4k_2) - 2\,z_x\right)\\
&\times [2k_1 - v]^{-1}.
\end{aligned} \tag{5.8}$$

Then we differentiate (5.2) with respect to t ($s(t) = 1$)

$$\begin{aligned}
v_t = &-\frac{1}{2}\left(-\frac{1}{2}\int u_t\,dx\,\mathrm{e}^{-1/2\int u(x,t)\,dx} - 2\,z_t\right)\\
&\times\left[\sqrt{4\,{k_1}^2 + \mathrm{e}^{-1/2\int u(x,t)\,dx} - 2z(x,t) + 4\,k_2}\right]^{-1}.
\end{aligned} \tag{5.9}$$

Inserting into this expression

$$\int u(x,t)\,dx = -\frac{1}{2}u^2 + u_x,$$

with taking into account (5.4) and applying (5.7), we obtain a conjugate equation for the wished BT

$$v_t = \left(-\frac{1}{2} \frac{-\frac{1}{2}\left(-\frac{1}{2}u^2 + u_x\right)\left(2\,z + v^2 - 4k_1 v - 4k_2\right) - 2\,z_x}{2k_1 - v} \right)^2 + z_{xx}. \tag{5.10}$$

A cross differentiation of obtained expressions (5.8), (5.10) generates a manifold defined by the equations (3.10) in the form (5.4) and (1.7).

On the other hand side, solving (5.5) with respect to z_x we find first component of another BT

$$z_x = u\left(-\frac{1}{2}z - \frac{1}{4}v^2 + k_1 v + k_2\right) - v_x(v - k_1). \tag{5.11}$$

Differentiating (5.11) with respect to x and substituting the result into (3.10) we obtain a conjugate equation for the searched BT

$$z_t = \left(u_x - \frac{1}{2}u^2\right)\left(-\frac{1}{2}z - \frac{1}{4}v^2 + k_1 v + k_2\right) - v_{x\,x}(v - 2k_1). \tag{5.12}$$

This enables one to construct by a cross differentiation of obtained expressions (5.11), (5.12) a manifold defined by the equations (1.8) and (1.7). Making transition onto the manifold defined by one equation one gets a differential consequences of another one.

6. Conclusion

The natural generalizations of a method of nonlocal transformations results in application of prolonged nonlocal transformations to the prolonged PDEs that do not admit the direct connection under appropriate nonlocal transformations in one step. It appear to be a unifying concept for nonlocal transformations for searching symmetries and solutions of nonlinear DEs. A method of construction of adjoint solution to the initial equation and a new concept of nonlocal transformation with additional variables which are introduced, developed and used in present paper, result in discovery of new approach for usage of conservation laws and for understanding the relations between different known solution techniques. A problem of inversion of the nonlocal transformation with additional variables is discussed. Several examples are presented. Derived technique is applied for construction of the formulae of generation of solutions. This proposed additional nonlocal symmetry allows for the generation of more wide families of exact solutions to nonlinear differential equations is called forced symmetries. The formulae derived are used for construction of exact solutions of some nonlinear equations.

We hope that the methods presented in this paper can be applied to different classes of nonlinear equations. All results obtained in the present paper for the considered equations can be extended to similar classes of equations via application of appropriate point transformations.

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