Проблеми математичного моделювання
та теорії диференціальних рівнянь

УДК 519.6

ON APPROXIMATE SOLUTIONS TO ONE CLASS OF NON-LINEAR DIRICHLET ELLIPTIC BOUNDARY VALUE PROBLEMS

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Communicated by Prof. T. Horsin

We discuss the existence of weak solutions to one class of Dirichlet boundary value problems (BVP) for non-linear elliptic equation. Because of the specific of nonlinearity, we cannot a priori expect to have a solution in the standard functional space. Instead of this we show that the original BVP admits the so-called approximate weak solutions. To do so, we introduce a special family of perturbed optimal control problems (OCPs). The main question we discuss in this paper is about solvability of perturbed OCPs, uniqueness of their solutions, and asymptotic properties of optimal pairs as the perturbation parameter $\varepsilon > 0$ tends to zero. As a result, we derive the sufficient conditions of the existence of weak solutions to the given class of nonlinear Dirichlet BVP and give a practical way for the approximation of such solutions.

Keywords: existence result, elliptic equations, fictitious control, perturbation approach

1. Introduction

In this paper we are concerned with the following Dirichlet boundary value problem

$$-\Delta y = F'(y) + g \quad \text{in} \ \ \Omega, \tag{1.1}$$

$$y = 0 \quad \text{on} \ \ \partial\Omega, \tag{1.2}$$

where $F \in C^1_{loc}(\mathbb{R})$, $F(z) \geq \exp(C_F z)$ for all $z \in \mathbb{R}$, and $g \in L^p(\Omega)$, $2 < p < \infty$, is a given distribution.

It is well known that the indicated BVP is ill-posed in general. It means that there is no reason to assert the existence of weak solutions to (1.1)–(1.2) for a given $g \in L^p(\Omega)$, or to suppose that such solution, even if it exists, is unique (see, for instance, I.M. Gelfand [11], M.G. Crandall and P.H. Rabinowitz

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[5], F. Mignot and J.P. Puel [15], Th. Gallouët, F. Mignot and J.P. Puel [10], H. Fujita [8], R.G. Pinsky [16], R. Ferreira, A. De Pablo, J.L. Vazquez [6]). At the same time, the BVPs like (1.1)–(1.2) appear in many contexts: in the study of stellar structures [4], in combustion theory for the chemical reactors [7], and etc.

The aim of this article is to discuss the existence of weak solutions to the Dirichlet boundary value problem (1.1)–(1.2). Because of the specific of non-linearity $F(y)$, we cannot assert that the indicated BVP admits at least one solution in the standard functional space. Instead of this we show that the original BVP possesses the so-called approximate weak solutions. We define these solutions as the weak solution to the problem (1.1)–(1.2) with special choice of the distribution $g^* \in \overline{\Lambda}$ which must be close (in some sense) to the original g. Since the set of feasible distributions $\overline{\Lambda}$ has rather complicate structure, it is not easy matter to touch on the choice of $g^* \in \overline{\Lambda}$ directly. In view of this, we introduce a special family of perturbed optimal control problems (OCPs) $\left\langle \inf_{(u,y) \in \Xi_\Delta} J_\varepsilon(u, y) \right\rangle$, where

$$J_\varepsilon(u, y) = \frac{\varepsilon}{2} \|\Delta y\|_{L^2(\Omega)}^2 + \frac{1}{p} \|g - u\|_{L^p(\Omega)}^p$$

and the functional properties of fictitious controls u are closely related with the properties of the distribution g in right-hand side of the elliptic equation.

The main questions we discuss in this paper are about solvability of perturbed OCPs, uniqueness of their solutions, and asymptotic properties of optimal pairs as the perturbation parameter $\varepsilon > 0$ tends to zero. As a result, we show that the optimal pairs $\left\{(u_\varepsilon^0, y_\varepsilon^0)\right\}_{\varepsilon>0} \subset L^p(\Omega) \times H_0^1(\Omega)$ can be defined in a unique way and each weakly cluster point of this sequence has a structure (g^*, y_g), where y_g stands for the approximate solution to the original BVP. Thus, we derive the sufficient conditions of the existence of weak solutions to the given class of nonlinear Dirichlet BVP and give a practical way for the approximation of such solutions (for the details we refer to Theorem 5.2).

2. Statement of the Problem and Some Preliminaries

Let Ω be a bounded open connected subset of $\mathbb{R}^N$ ($N > 2$). We assume that the boundary $\partial\Omega$ is Lipschitzian so that the unit outward normal $\nu = \nu(x)$ is well-defined for a.e. $x \in \partial\Omega$, where a.e. means here with respect to the $(N-1)$-dimensional Hausdorff measure. We also assume that Ω is star-shaped with respect to some interior point x_0, i.e. $(\sigma - x_0, \nu(\sigma)) \geq 0$ for almost all $\sigma \in \partial\Omega$.

Let $F : \mathbb{R} \to (0, +\infty)$ be a mapping such that $F \in C_{loc}^1(\mathbb{R})$ and there exists a constant $C_F > 0$ satisfying

$$F(z) \leq C_F F'(z), \quad \forall z \in \mathbb{R}, \tag{2.1}$$

that is, in what follows we may suppose that $F(z) \geq \exp(C_F z)$ over $\mathbb{R}$. Let p ($2 \leq p < \infty$) be a real number and let $g \in L^p(\Omega)$ be a given distribution. By

$H_0^1(\Omega)$ we denote the Sobolev space as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|y\|_{H_0^1(\Omega)} = \left(\int_\Omega |\nabla y|^2\, dx\right)^{1/2}$. Let $H^{-1}(\Omega)$ be the dual space to $H_0^1(\Omega)$.

In order to make a precise meaning of the solution to BVP (1.1)–(1.2) and indicate its characteristic properties, we begin with the following concept.

Definition 2.1. We say that a function $y = y(g)$ is a weak solution to the boundary value problem (1.1)–(1.2) for a given distribution $g \in L^p(\Omega)$ if $y \in H_0^1(\Omega)$, y belongs to the class of functions

$$Y = \left\{ y \in H_0^1(\Omega) \;\middle|\; F'(y) \in L^1(\Omega) \right\}, \tag{2.2}$$

and the integral identity

$$\int_\Omega (\nabla y, \nabla \varphi)\, dx = \int_\Omega F'(y)\varphi\, dx + \int_\Omega g\varphi\, dx \tag{2.3}$$

holds for every test function $\varphi \in C_0^\infty(\Omega)$.

As was indicated before, it is unknown whether the original BVP admits at least one weak solution in the sense of Definition 2.1 for a given distribution $g \in L^p(\Omega)$. Moreover, as follows from (2.2), the continuity of form $[y, \varphi]_F := \int_\Omega F'(y)\varphi\, dx$ on the set $Y \subset H_0^1(\Omega)$ is not evident. This motivates us to introduce of the following set.

Definition 2.2. We say that an element $y \in H_0^1(\Omega)$ belongs to the set H_F if

$$\left| \int_\Omega F'(y)\varphi\, dx \right| \le c(y) \left(\int_\Omega |\nabla \varphi|^2\, dx \right)^{1/2}, \quad \forall \varphi \in C_0^\infty(\Omega) \tag{2.4}$$

with some constant depending on y.

As a result, we have: if $y \in H_F$ then the mapping $\varphi \mapsto [y, \varphi]_F$ can be defined for all $\varphi \in H_0^1(\Omega)$ using (2.4) and the standard rule

$$[y, \varphi]_F = \lim_{\varepsilon \to 0} [y, \varphi_\varepsilon]_F, \tag{2.5}$$

where $\{\varphi_\varepsilon\}_{\varepsilon > 0} \subset C_0^\infty(\Omega)$ and $\varphi_\varepsilon \to \varphi$ strongly in $H_0^1(\Omega)$. In particular, if $y \in H_F$, then we can define the value $[y, y]_F$ and this one is finite for every $y \in H_F$, although the 'integrand' $yF'(y)$ needs not to be integrable on Ω, in general. Taking this fact into account, we immediately arrive at the following conclusion.

Proposition 2.1. If $g \in L^p(\Omega)$ is a given distribution and $y \in H_F$ is a weak solution to BVP (1.1)–(1.2) in the sense of Definition (2.1), then y satisfies the energy equality

$$\int_\Omega |\nabla y|^2\, dx = [y, y]_F + \int_\Omega gy\, dx. \tag{2.6}$$

We note that by the initial assumptions and Holder's inequality, this relation makes a sense because

$$\left|\int_\Omega g y \, dx\right| \leq \|g\|_{L^2(\Omega)} \|y\|_{L^2(\Omega)} \leq |\Omega|^{\frac{p-2}{2p}} \|g\|_{L^p(\Omega)} C_\Omega \|y\|_{H_0^1(\Omega)} < +\infty. \quad (2.7)$$

However, since the value $[y,y]_F$ does not preserve a constant sign for all $y \in H_F$, it follows that we cannot make use of the energy equality (2.6) in order to derive a reasonable a priori estimate in $\|\cdot\|_{H_0^1(\Omega)}$-norm for the weak solutions. In particular, to specify the term $[y,y]_F$ we can use the following result.

Lemma 2.1. *Let $y \in Y$ be a weak solution to BVP (1.1)–(1.2). Then $y \in H_F$, $F'(y) \in H^{-1}(\Omega)$, and*

$$[y,z]_F = \left\langle F'(y), z \right\rangle_{H^{-1}(\Omega); H_0^1(\Omega)} = \int_\Omega z \, F'(y) \, dx, \quad \forall z \in H_0^1(\Omega), \quad (2.8)$$

i.e. $z \, F'(y) \in L^1(\Omega)$ for every $z \in H_0^1(\Omega)$.

Remark 2.1. Here, by $H^{-1}(\Omega)$ we mean the completeness of the Hilbert space $L^2(\Omega)$ with respect to the norm

$$\|f\|_{H^{-1}(\Omega)} = \sup_{\varphi \in H_0^1(\Omega), \varphi \neq 0} \left[\int_\Omega f(x) \varphi(x) \, dx \left(\|\varphi\|_{H_0^1(\Omega)} \right)^{-1} \right], \quad \forall f \in L^2(\Omega). \quad (2.9)$$

Proof. Following the definition of the weak solution, we have (see (2.3))

$$\int_\Omega F'(y) \varphi \, dx \leq \left| \int_\Omega (\nabla y, \nabla \varphi) \, dx \right| + \left| \int_\Omega g \varphi \, dx \right|$$

$$\leq \|y\|_{H_0^1(\Omega)} \left(\int_\Omega |\nabla \varphi|^2 \, dx \right)^{1/2} + \|g\|_{L^2(\Omega)} \|\varphi\|_{L^2(\Omega)}$$

$$\overset{\text{by } (2.7)}{\leq} \left(\|y\|_{H_0^1(\Omega)} + C_\Omega |\Omega|^{\frac{p-2}{2p}} \|g\|_{L^p(\Omega)} \right) \|\varphi\|_{H_0^1(\Omega)}, \quad \forall \varphi \in C_0^\infty(\Omega). \quad (2.10)$$

Hence, $y \in H_F$ by Definition 2.2.

Let $z \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be an arbitrary element. Since $F'(y) \in L^1(\Omega)$, it follows that the term $\int_\Omega z \, F'(y) \, dx$ is well defined. Let $\{\varphi_\varepsilon\}_{\varepsilon > 0} \subset C_0^\infty(\Omega)$ be a sequence such $\varphi_\varepsilon \to z$ in $H_0^1(\Omega)$. In view of our assumptions, it is plausible to suppose that

$$\sup_{\varepsilon > 0} \|\varphi_\varepsilon\|_{L^\infty(\Omega)} < +\infty \quad \text{and} \quad \varphi_\varepsilon \overset{*}{\rightharpoonup} z \text{ in } L^\infty(\Omega).$$

Hence, using the fact that $y \in H_F$, we get

$$\int_\Omega z \, F'(y) \, dx = \lim_{\varepsilon \to 0} \int_\Omega \varphi_\varepsilon F'(y) \, dx = \lim_{\varepsilon \to 0} [y, \varphi_\varepsilon]_F \overset{\text{by } (2.5)}{=} [y, z]_F. \quad (2.11)$$

Thus, we arrive at the relation (2.8) for each $z \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

Let us take now $z \in H_0^1(\Omega)$ such that $z \geq 0$ almost everywhere in Ω. For every $\varepsilon > 0$, let $T_\varepsilon : \mathbb{R} \to \mathbb{R}$ be the truncation operator defined by

$$T_\varepsilon(s) = \max\left\{\min\left\{s, \varepsilon^{-1}\right\}, -\varepsilon^{-1}\right\} \tag{2.12}$$

It is well-know the following property of T_ε [13]: If $z \in H_0^1(\Omega)$ then

$$T_\varepsilon(z) \in L^\infty(\Omega) \cap H_0^1(\Omega) \ \forall \varepsilon > 0 \quad \text{and} \quad T_\varepsilon(z) \to z \ \text{ in } H_0^1(\Omega) \text{ as } \varepsilon \to 0.$$

Hence, $T_\varepsilon(z) \to z$ almost everywhere in Ω. Since

$$T_\varepsilon(z)F'(y) \overset{\text{by } (2.1)}{\geq} \frac{1}{C_F}T_\varepsilon(z)F(y) > 0 \quad \text{in } \Omega \tag{2.13}$$

$\{T_\varepsilon(z)F'(y)\}_{\varepsilon>0}$ is a pointwise non-decreasing sequence, and $T_\varepsilon(z)F'(y) \to z\,F'(y)$ for almost all $x \in \Omega$, it follows by monotone convergence theorem that $z\,F'(y)$ is a measurable function on Ω and

$$\lim_{\varepsilon \to 0} \int_\Omega T_\varepsilon(z)F'(y)\,dx = \int_\Omega z\,F'(y)\,dx.$$

Hence, the relation (2.13) remains valid. Thus, (2.8) holds true for each $z \in H_0^1(\Omega)$ such that $z \geq 0$.

As for a general case, i.e. $z \in H_0^1(\Omega)$, it is enough to note that $z = z^+ - z^-$ with $z^+, z^- \in H_0^1(\Omega)$ and $z^+, z^- \geq 0$ in Ω, where

$$z^+ := \max\left\{z, 0\right\}, \quad z^- := \max\left\{-z, 0\right\}.$$

To complete the proof, it remains to observe that

$$\int_\Omega z\,F'(y)\,dx \overset{\text{by } (2.13)}{=} \lim_{\varepsilon \to 0} \int_\Omega \varphi_\varepsilon F'(y)\,dx$$

$$\overset{\text{by } (2.10)}{\leq} \lim_{\varepsilon \to 0}\left(\|y\|_{H_0^1(\Omega)} + C_\Omega|\Omega|^{\frac{p-2}{2p}}\|g\|_{L^p(\Omega)}\right)\|\varphi_\varepsilon\|_{H_0^1(\Omega)}$$

$$\text{(by the strong convergence of } \varphi_\varepsilon \to z \text{ in } H_0^1(\Omega))$$

$$= \left(\|y\|_{H_0^1(\Omega)} + C_\Omega|\Omega|^{\frac{p-2}{2p}}\|g\|_{L^p(\Omega)}\right)\|z\|_{H_0^1(\Omega)}$$

holds true for an arbitrary element $z \in H_0^1(\Omega)$. As a result, we have

$$F'(y) \in H^{-1}(\Omega), \quad \left\langle F'(y), z\right\rangle_{H^{-1}(\Omega);H_0^1(\Omega)} = \int_\Omega z\,F'(y)\,dx, \quad \forall z \in H_0^1(\Omega),$$

$$\text{and} \quad \|F'(y)\|_{H^{-1}(\Omega)} \leq \left(\|y\|_{H_0^1(\Omega)} + C_\Omega|\Omega|^{\frac{p-2}{2p}}\|g\|_{L^p(\Omega)}\right),$$

where the constant C_Ω comes from the Friedrichs–Poincaré inequality. $\qquad\square$

Remark 2.2. To clarify the main conclusion of Lemma 2.1, we make use of the following observations (see Remark 2.1). Let $\{f_k\}_{k \in \mathbb{N}} \subset L^2(\Omega)$ be a Cauchy sequence with respect to the norm $\| \cdot \|_{H^{-1}(\Omega)}$. Then, following the definition of the norm $\| \cdot \|_{H^{-1}(\Omega)}$, we have

$$\int_\Omega f_k(x)\varphi(x)\,dx \leq \|f_k\|_{H^{-1}(\Omega)}\|\varphi\|_{H_0^1(\Omega)}.$$

Hence, the H^{-1}-limit of the sequence $\{f_k\}_{k \in \mathbb{N}}$ is a linear functional f on $H_0^1(\Omega)$ satisfying the estimate

$$\left| <f,\varphi>_{(H_0^1(\Omega))^*;H_0^1(\Omega)} \right| \leq \|f\|_{H^{-1}(\Omega)}\|\varphi\|_{H_0^1(\Omega)}, \quad \forall \varphi \in H_0^1(\Omega),$$

that is, $\lim_{k\to\infty} \int_\Omega f_k(x)\varphi(x)\,dx = <f,\varphi>_{(H_0^1(\Omega))^*;H_0^1(\Omega)}$ and $f \in \left(H_0^1(\Omega)\right)^*$. In order to show that $f \in H^{-1}(\Omega)$, we make use of the Riesz Representation Theorem. As a result, for a given $f \in \left(H_0^1(\Omega)\right)^*$ there exists an element $u_f \in H_0^1(\Omega)$ such that

$$<f,\varphi>_{(H_0^1(\Omega))^*;H_0^1(\Omega)} = \int_\Omega (\nabla u_f(x), \nabla\varphi(x))\ dx, \quad \forall\,\varphi \in H_0^1(\Omega).$$

By definition of the generalized derivative, we have

$$\begin{aligned}
(-\Delta u_f)[\varphi] &= \int_\Omega (\nabla u_f(x), \nabla\varphi(x))\ dx \\
&\leq \|\nabla u_f\|_{L^2(\Omega)^N}\|\nabla\varphi\|_{L^2(\Omega)^N} = \|u_f\|_{H_0^1(\Omega)}\|\varphi\|_{H_0^1(\Omega)}
\end{aligned} \tag{2.14}$$

for all $\varphi \in C_0^\infty(\Omega)$. Since $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, we can infer from (2.14) and (2.9) that

$$\| -\Delta u_f\|_{H^{-1}(\Omega)} = \sup_{\varphi \in H_0^1(\Omega), \varphi \neq 0} \frac{\int_\Omega (\nabla u_f(x), \nabla\varphi(x))\ dx}{\|\varphi\|_{H_0^1(\Omega)}} \leq \|u_f\|_{H_0^1(\Omega)}.$$

Hence, the linear operator $-\Delta : H_0^1(\Omega) \mapsto H^{-1}(\Omega)$ is continuous. Moreover, this operator is surjective by Riesz Representation Theorem. Thus, $-\Delta : H_0^1(\Omega) \mapsto H^{-1}(\Omega)$ is a bijective bounded operator and, therefore, the inverse mapping theorem implies that $-\Delta$ is an isomorphism between Banach spaces $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. In view of this, the spaces $\left(H_0^1(\Omega)\right)^*$ and $H^{-1}(\Omega)$ are isomorphic as well. Finally, it means that

$$\lim_{k\to\infty} \int_\Omega f_k(x)\varphi(x)\,dx = <f,\varphi>_{H^{-1}(\Omega);H_0^1(\Omega)} \quad \text{and} \quad f \in H^{-1}(\Omega).$$

As a direct consequence of this lemma and Proposition 2.1, we have the following result.

Corollary 2.1. *Let $g \in L^p(\Omega)$ be a given distribution and let $y \in H_0^1(\Omega)$ be a weak solution to BVP (1.1)–(1.2) in the sense of Definition (2.1). Then the energy equality for y takes the form*

$$\int_\Omega |\nabla y|^2 \, dx = \int_\Omega y \, F'(y) \, dx + \int_\Omega gy \, dx. \qquad (2.15)$$

3. Some Auxiliary Results and a Priori Estimates

In this Section we deal with some extra properties of the weak solutions to the boundary value problem (1.1)–(1.2). In many aspects we follow the ideas of the paper E. Casas, O. Kavian, and J.P. Puel [3].

Proposition 3.1. *Let $y \in H_0^1(\Omega)$ be a weak solution to BVP (1.1)–(1.2) such that $\Delta y \in L^2(\Omega)$. Then*

$$\left(\frac{N}{2} - 1 \right) \int_\Omega |\nabla y|^2 \, dx \le N \int_\Omega \left(F(y) - F(0) \right) \, dx - \int_\Omega g \left(x - x_0, \nabla y \right) \, dx, \quad (3.1)$$

where $x_0 \in \operatorname{int} \Omega$ is a point such that $(\sigma - x_0, \nu(\sigma)) \ge 0$ for almost all $\sigma \in \partial\Omega$, and $\nu(\sigma)$ denotes the outward unit normal vector to $\partial\Omega$ at the point σ.

Proof. In view of the initial assumptions, we have $-\Delta y - g \in L^2(\Omega)$. Hence, $F'(y) \in L^2(\Omega)$ and, therefore, we can multiplay the equation (1.1) by any function $\varphi \in L^2(\Omega)$. Let us take $\varphi := (x - x_0, \nabla y) \in L^2(\Omega)$. Then

$$\int_\Omega \left[-\Delta y \left(x - x_0, \nabla y \right) \right] dx = \int_\Omega F'(y) \left(x - x_0, \nabla y \right) dx + \int_\Omega g \left(x - x_0, \nabla y \right) dx. \tag{3.2}$$

Step 1. We apply the formula of integration by parts to the left hand side of (3.2). This yields

$$\int_\Omega \left[-\Delta y \left(x - x_0, \nabla y \right) \right] dx = \int_\Omega \left(\nabla y, \nabla \left(x - x_0, \nabla y \right) \right) dx$$

$$- \int_{\partial\Omega} \frac{\partial y(\sigma)}{\partial \nu} \left(\sigma - x_0, \nabla y(\sigma) \right) d\sigma = \sum_{i=1}^N \int_\Omega \frac{\partial y}{\partial x_i} \frac{\partial}{\partial x_i} \left[\sum_{j=1}^N (x_j - x_{0j}) \frac{\partial y}{\partial x_j} \right] dx$$

$$- \int_{\partial\Omega} \frac{\partial y(\sigma)}{\partial \nu} \left(\sigma - x_0, \nabla y(\sigma) \right) d\sigma = \sum_{i=1}^N \int_\Omega \left| \frac{\partial y}{\partial x_i} \right|^2 dx$$

$$+ \sum_{i,j=1}^N \int_\Omega (x_j - x_{0j}) \frac{\partial y}{\partial x_i} \frac{\partial^2 y}{\partial x_i \partial x_j} dx - \int_{\partial\Omega} \frac{\partial y(\sigma)}{\partial \nu} \left(\sigma - x_0, \nabla y(\sigma) \right) d\sigma$$

$$= \int_\Omega |\nabla y|^2 \, dx + \frac{1}{2} \sum_{i,j=1}^N \int_\Omega (x_j - x_{0j}) \frac{\partial}{\partial x_j} \left(\frac{\partial y}{\partial x_i} \right)^2 \, dx$$

$$- \int_{\partial\Omega} \frac{\partial y(\sigma)}{\partial \nu} (\sigma - x_0, \nabla y(\sigma)) \, d\sigma = \int_\Omega |\nabla y|^2 \, dx$$

$$+ \frac{1}{2} \int_{\partial\Omega} \sum_{j=1}^N [(\sigma_j - x_{0j})\nu_j(\sigma)] \left[\sum_{i=1}^N \left(\frac{\partial y(\sigma)}{\partial \sigma_i} \right)^2 \right] d\sigma - \frac{1}{2} \sum_{j=1}^N \int_\Omega \left[\sum_{i=1}^N \left(\frac{\partial y}{\partial x_i} \right)^2 \right] dx$$

$$- \int_{\partial\Omega} \frac{\partial y(\sigma)}{\partial \nu} (\sigma - x_0, \nabla y(\sigma)) \, d\sigma = \left(1 - \frac{N}{2} \right) \int_\Omega |\nabla y|^2 \, dx$$

$$+ \frac{1}{2} \int_{\partial\Omega} (\sigma - x_0, \nu(\sigma)) |\nabla y(\sigma)|^2 \, d\sigma - \int_{\partial\Omega} \frac{\partial y(\sigma)}{\partial \nu} (\sigma - x_0, \nabla y(\sigma)) \, d\sigma.$$

Since for $\sigma \in \partial\Omega$ we have $\nabla y(\sigma) = \pm |\nabla y(\sigma)| \nu(\sigma)$, it follows that

$$(\sigma - x_0, \nabla y(\sigma)) = \pm |\nabla y(\sigma)| (\sigma - x_0, \nu(\sigma)),$$

$$\frac{\partial y(\sigma)}{\partial \nu} = (\nabla y(\sigma), \nu(\sigma)) = \pm |\nabla y(\sigma)|.$$

Hence,

$$\int_{\partial\Omega} \frac{\partial y(\sigma)}{\partial \nu} (\sigma - x_0, \nabla y(\sigma)) \, d\sigma = \int_{\partial\Omega} (\sigma - x_0, \nu(\sigma)) |\nabla y(\sigma)|^2 \, d\sigma,$$

and, therefore,

$$\int_\Omega [-\Delta y \, (x - x_0, \nabla y)] \, dx$$

$$= \left(1 - \frac{N}{2} \right) \int_\Omega |\nabla y|^2 \, dx - \frac{1}{2} \int_{\partial\Omega} (\sigma - x_0, \nu(\sigma)) |\nabla y(\sigma)|^2 \, d\sigma$$

$$\underset{\substack{\text{by the star-shaped} \\ \text{properties of } \Omega}}{\geq} \left(1 - \frac{N}{2} \right) \int_\Omega |\nabla y|^2 \, dx. \tag{3.3}$$

Step 2. Before proceeding further, let us show that the following relation

$$\int_\Omega F'(y) (\nabla y, \psi) \, dx = - \int_\Omega [F(y) - F(0)] \operatorname{div} \psi \, dx \tag{3.4}$$

holds true for any vector-valued test function $\psi \in C^1(\overline{\Omega})^N$ provided $y \in H_0^1(\Omega)$ is a weak solution to (1.1)–(1.2). Indeed, let $T_\varepsilon : \mathbb{R} \to \mathbb{R}$ be the truncation operator defined in (2.12). By definition of T_ε, we have

$$T_\varepsilon(y) \to y \text{ strongly in } H_0^1(\Omega) \text{ and almost everywhere in } \Omega \text{ as } \varepsilon \to 0. \tag{3.5}$$

Moreover, since $F'(y) \in L^2(\Omega)$ and $F' \in C_{loc}(\mathbb{R})$, it follows from (3.5) that

$$F'(T_\varepsilon(y)) \to F'(y) \quad \text{in } L^2(\Omega) \text{ and almost everywhere in } \Omega.$$

Then Lebesgue Dominated Theorem implies: $F'(T_\varepsilon(y)) \nabla T_\varepsilon(y) \to F'(y)\nabla y$ in $L^1(\Omega)^N$. Taking into account the fact that

$$F'(T_\varepsilon(y)) \nabla T_\varepsilon(y) = \nabla F(T_\varepsilon(y)), \quad \forall \varepsilon > 0,$$

we conclude

$$F'(y)\nabla y = \nabla (F(y) - F(0)) \quad \text{as elements of } L^1(\Omega).$$

As a result, the equality (3.4) is a direct consequence of the formula of integration by parts.

Now we are in a position to transform of the right hand side in (3.2). Indeed, due to relation (3.4), we have

$$\int_\Omega F'(y)(x - x_0, \nabla y) \, dx = \int_\Omega (x - x_0, \nabla [F(y) - F(0)]) \, dx$$
$$\overset{\text{by } (3.4)}{=} -\int_\Omega [F(y) - F(0)] \operatorname{div}(x - x_0) \, dx = -N \int_\Omega [F(y) - F(0)] \, dx.$$

Combining this equality with (3.2) and inequality (3.3), we arrive at the desired relation (3.1). The proof is complete. □

The next result is crucial in this section. Namely, we show that inequality (3.1) implies some a priori estimate for the weak solutions $y \in Y$ to the original BVP.

Theorem 3.1. *Let $y \in Y$ be a weak solution to BVP (1.1)–(1.2) such that y satisfies the inequality (3.1). Then*

$$\int_\Omega y \, F'(y) \, dx \le C_1 \|g\|_{L^p(\Omega)}^2 + C_2 \|g\|_{L^p(\Omega)} + C_3, \tag{3.6}$$

$$\|y\|_{H_0^1(\Omega)} \le C_4 \|g\|_{L^p(\Omega)} + C_5, \tag{3.7}$$

for some positive constants C_i, $1 \le i \le 5$, independent of g and y.

Proof. Combining the energy equality (2.6) with inequality (3.1), we get

$$\left(\frac{N}{2} - 1\right) \int_\Omega y \, F'(y) \, dx + \left(\frac{N}{2} - 1\right) \int_\Omega gy \, dx$$
$$\le N \int_\Omega (F(y) - F(0)) \, dx - \int_\Omega g(x - x_0, \nabla y) \, dx.$$

Hence, in view of estimate (2.7), we can rewrite the last relation as follows

$$\int_\Omega y \, F'(y) \, dx \le \frac{2N}{N-2} \int_\Omega (F(y) - F(0)) \, dx$$
$$+ |\Omega|^{\frac{p-2}{2p}} \frac{2C_\Omega \operatorname{diam} \Omega}{N-2} \|g\|_{L^p(\Omega)} \|y\|_{H_0^1(\Omega)}. \tag{3.8}$$

For our further analysis, we set

$$\Omega_N := \left\{ x \in \Omega \; : \; y(x) > \frac{4NC_F}{N-2} \right\},$$

where the constant C_F is defined in (2.1). Since $F : \mathbb{R} \to (0, +\infty)$, it follows from (2.1) that

$$\frac{2N}{N-2} \int_\Omega (F(y) - F(0)) \, dx$$
$$\leq \frac{2N}{N-2} \int_\Omega F(y) \, dx \leq \frac{2NC_F}{N-2} \int_\Omega F'(y) \, dx$$
$$\leq \frac{1}{2} \int_{\Omega_N} y F'(y) \, dx + \frac{2NC_F}{N-2} \int_{\Omega \setminus \Omega_N} F' \left(\frac{4NC_F}{N-2} \right) \, dx$$
$$\leq \frac{1}{2} \int_{\Omega_N} y F'(y) \, dx + \frac{2N}{N-2} C_F |\Omega| F' \left(\frac{4NC_F}{N-2} \right)$$

and

$$\int_{\Omega \setminus \Omega_N} y F'(y) \, dx \leq \frac{4NC_F}{N-2} \int_{\Omega \setminus \Omega_N} F' \left(\frac{4NC_F}{N-2} \right) \, dx$$
$$\leq \frac{4N}{N-2} C_F |\Omega| F' \left(\frac{4NC_F}{N-2} \right). \tag{3.9}$$

Then inequality (3.8) yields to the following relation

$$\int_{\Omega_N} y \, F'(y) \, dx = \int_\Omega y \, F'(y) \, dx - \int_{\Omega \setminus \Omega_N} y \, F'(y) \, dx$$
$$\overset{\text{by (3.8)}}{\leq} \frac{1}{2} \int_{\Omega_N} y F'(y) \, dx + \frac{2N}{N-2} C_F |\Omega| F' \left(\frac{4NC_F}{N-2} \right) - \int_{\Omega \setminus \Omega_N} y \, F'(y) \, dx$$
$$+ |\Omega|^{\frac{p-2}{2p}} \frac{2C_\Omega \operatorname{diam} \Omega}{N-2} \|g\|_{L^p(\Omega)} \|y\|_{H_0^1(\Omega)}.$$

Therefore,

$$\frac{1}{2} \int_\Omega y \, F'(y) \, dx \leq \frac{2N}{N-2} C_F |\Omega| F' \left(\frac{4NC_F}{N-2} \right) - \frac{1}{2} \int_{\Omega \setminus \Omega_N} y \, F'(y) \, dx$$
$$+ |\Omega|^{\frac{p-2}{2p}} \frac{2C_\Omega \operatorname{diam} \Omega}{N-2} \|g\|_{L^p(\Omega)} \|y\|_{H_0^1(\Omega)}.$$

As a result, we get from (3.9) and the previous inequality

$$\int_\Omega y \, F'(y) \, dx \leq \frac{8N}{N-2} C_F |\Omega| F' \left(\frac{4NC_F}{N-2} \right)$$
$$+ 4|\Omega|^{\frac{p-2}{2p}} \frac{C_\Omega \operatorname{diam} \Omega}{N-2} \|g\|_{L^p(\Omega)} \|y\|_{H_0^1(\Omega)}$$
$$= \widehat{C}_1 + 2\widehat{C}_2 \|g\|_{L^p(\Omega)} \|y\|_{H_0^1(\Omega)}. \tag{3.10}$$

Finally using the energy equality (2.6), we obtain

$$\|y\|^2_{H^1_0(\Omega)} \leq \widehat{C}_1 + 3\widehat{C}_2 \|g\|_{L^p(\Omega)} \|y\|_{H^1_0(\Omega)}$$

and this implies the desired estimate (3.7). In order to establish the estimate (3.6), it is enough to make use of (3.7) in (3.10). The proof is complete. $\square$

Remark 3.1. It is worth to notice that inequality (3.1) makes sense even if we do not assume the fulfillment of inclusion $\Delta y \in L^2(\Omega)$ but have only that $y \in Y \subset H^1_0(\Omega)$ and $g \in L^p(\Omega)$. At the same time it is unknown whether this inequality holds for an arbitrary weak solution to BVP (1.1)–(1.2). Since the existence and uniqueness of the weak solutions to the original BVP is an open question for arbitrary given distribution $g \in L^p(\Omega)$, the following result reflects some interesting properties of weak solutions satisfying inequality (3.1).

Proposition 3.2. Let (g, y) be a given pair in $L^p(\Omega) \times H^1_0(\Omega)$ with $p > 2$. Let $\{(g_k, y_k)\}_{k \in \mathbb{N}} \subset L^p(\Omega) \times Y$ be a sequence such that, for each $k \in \mathbb{N}$ the pairs (g_k, y_k) are related by the integral identity (2.3), satisfy inequality (3.1), and

$$(g_k, y_k) \rightharpoonup (g, y) \quad \text{weakly in} \quad L^p(\Omega) \times H^1_0(\Omega) \quad \text{as } k \to \infty. \tag{3.11}$$

Then y is a weak solution to BVP (1.1)–(1.2) for given $g \in L^p(\Omega)$, the pair (g, y) satisfies the inequality (3.1), and

$$F'(y_k) \to F'(y) \quad \text{in} \quad L^1(\Omega) \quad \text{as } k \to \infty. \tag{3.12}$$

Proof. By the Rellich–Kondrachov theorem, the embedding $H^1_0(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Hence, the weak convergence $y_k \rightharpoonup y$ in $H^1_0(\Omega)$ implies the strong convergence in $L^2(\Omega)$. Therefore, up to a subsequence, we can suppose that $y_k(x) \to y(x)$ for almost every point $x \in \Omega$. As a result, we have the pointwise convergence: $F'(y_k) \to F'(y)$ everywhere in Ω. Let us show that this implies the strong convergence (3.12).

With that in mind we recall that a sequence $\{f_k\}_{k \in \mathbb{N}}$ is called equi-integrable on Ω if for any $\delta > 0$, there is a $\tau = \tau(\delta)$ such that $\int_S |f_k|\, dx < \delta$ for every measurable subset $S \subset \Omega$ of Lebesgue measure $|S| < \tau$. Let us show that the sequence $\{F'(y_k)\}_{k \in \mathbb{N}}$ is equi-integrable on Ω. To do so, we take $m > 0$ such that

$$m > \frac{2\left(C_1 \sup_{k \in \mathbb{N}} \|g_k\|^2_{L^p(\Omega)} + C_2 \sup_{k \in \mathbb{N}} \|g_k\|_{L^p(\Omega)} + C_3\right)}{\delta}. \tag{3.13}$$

We also set $\tau = \delta/(2F'(m))$. Then for every measurable set $S \subset \Omega$ with $|S| < \tau$, we have

$$\int_S F'(y_k)\, dx \leq \int_{\{x \in S \,:\, y_k(x) > m\}} F'(y_k)\, dx + \int_{\{x \in S \,:\, y_k(x) \leq m\}} F'(y_k)\, dx$$

$$\leq \frac{1}{m} \int_{\{x \in S \,:\, y_k(x) > m\}} y_k F'(y_k)\, dx + \int_{\{x \in S \,:\, y_k(x) \leq m\}} F'(m)\, dx$$

$$\overset{\text{by (3.6)}}{\leq} \frac{C_1 \|g_k\|^2_{L^p(\Omega)} + C_2 \|g_k\|_{L^p(\Omega)} + C_3}{m} + F'(m)|S| \overset{\text{by (3.13)}}{\leq} \frac{\delta}{2} + \frac{\delta}{2}.$$

As a result, the assertion (3.12) is a direct consequence of Lebesgue's Convergence Theorem. Hence, $y \in Y$ and it is easy to show that the limit pair (g, y) is related by the integral identity (2.3). Indeed, in view of the initial assumptions and the strong convergence property (3.12), the limit passage in

$$\int_\Omega (\nabla y_k, \nabla \varphi)\, dx = \int_\Omega F'(y_k)\varphi\, dx + \int_\Omega g_k \varphi\, dx, \quad \forall\, \varphi \in C_0^\infty(\Omega)$$

becomes trivial. Thus, y is a weak solution to BVP (1.1)–(1.2) for given $g \in L^p(\Omega)$.

Our next intension is to prove that (g, y) satisfies (3.1). With that in mind we make use of the following result (see L. Boccardo and F. Murat [2]): if $y_k \rightharpoonup y$ in $H_0^1(\Omega)$ and the sequence $\{\Delta y_k\}_{k \in \mathbb{N}}$ is bounded in $L^1(\Omega)$, then, within a subsequence, $\nabla y_k(x) \to \nabla y(x)$ almost everywhere as $k \to \infty$. Indeed, as follows from (2.3),

$$\int_\Omega |\Delta y_k|\, dx \le \int_\Omega |F'(y_k)|\, dx + \int_\Omega |g_k|\, dx. \tag{3.14}$$

Hence, in view of (3.14) and $L^p(\Omega)$-boundedness of $\{g_k\}_{k \in \mathbb{N}}$ with $p > 2$, we have $\sup_{k \in \mathbb{N}} \|\Delta y_k\|_{L^1(\Omega)} < +\infty$. Thus, in what follows we may suppose that

$$\nabla y_k \rightharpoonup \nabla y \ \text{ in } \ L^2(\Omega)^N \ \text{ and } \ \nabla y_k(x) \rightharpoonup \nabla y(x) \text{ a.e. in } \ \Omega. \tag{3.15}$$

Let us show that this fact implies the strong convergence of gradients $\nabla y_k \to \nabla y$ in $L^{p'}(\Omega)^N$ with $p' = p/(1 - p)$. Indeed, for an arbitrary small set A, by the Hölder inequality for $1/q + 1/q' = 1$, we have

$$\int_A |\nabla y_k - \nabla y|^{p'}\, dx \le \left(\int_A |\nabla y_k - \nabla y|^{p'q}\, dx \right)^{1/q} \left(\int_A 1^{q'}\, dx \right)^{1/q'}.$$

Having chosen $q > 1$ such that $p'q = 2$, we obtain

$$\frac{1}{q'} = 1 - \frac{1}{q} = 1 - \frac{p'}{2} = \frac{2 - p'}{2} = \frac{p - 2}{2(p - 1)} \quad \text{or} \quad q' = \frac{2(p - 1)}{p - 2}.$$

Then

$$\int_A |\nabla y_k - \nabla y|^{p'}\, dx \le |A|^{\frac{p-2}{2(p-1)}} \sup_{k \in \mathbb{N}} \|\nabla y_k - \nabla y\|_{L^2(\Omega)^N}^{p'} \le C|A|^{\frac{p-2}{2(p-1)}},$$

that is, the sequence $\left\{ |\nabla y_k - \nabla y|^{p'} \right\}_{k \in \mathbb{N}}$ is equi-integrable. Combining this fact with (3.15), by Lebesgue Convergence Theorem, we conclude: $|\nabla y_k - \nabla y|^{p'} \to 0$ strongly in $L^1(\Omega)$, and, therefore,

$$\nabla y_k \to \nabla y \quad \text{strongly in } L^{p'}(\Omega)^N \text{ with } p' = p/(1 - p). \tag{3.16}$$

As a result, we get

$$\lim_{k\to\infty} \int_\Omega g_k\,(x - x_0, \nabla y_k)\,dx \overset{\text{by (3.16)}}{=} \int_\Omega g\,(x - x_0, \nabla y)\,dx$$

$$\text{(as product of weakly and strongly convergent sequences)},$$

$$\lim_{k\to\infty} \int_\Omega (F(y_k) - F(0))\,dx \overset{\text{by (3.12)}}{=} \int_\Omega (F(y) - F(0))\,dx,$$

$$\liminf_{k\to\infty} \int_\Omega |\nabla y|^2\,dx \overset{\text{by (3.11)}}{\geq} \int_\Omega |\nabla y|^2\,dx.$$

Then we can pass to the limit in the inequality (3.1) to finally obtain

$$\left(\frac{N}{2} - 1\right) \int_\Omega |\nabla y|^2\,dx \leq \left(\frac{N}{2} - 1\right) \liminf_{k\to\infty} \int_\Omega |\nabla y|^2\,dx$$

$$\leq \liminf_{k\to\infty} \left[N \int_\Omega (F(y_k) - F(0))\,dx - \int_\Omega g_k\,(x - x_0, \nabla y_k)\,dx\right]$$

$$= N \int_\Omega (F(y) - F(0))\,dx - \int_\Omega g\,(x - x_0, \nabla y)\,dx.$$

The proof is complete. $\qquad\qquad\square$

4. Fictitious Controls and Associated Optimal Control Problems

Let us consider the following sequence of optimal control problems (OCPs) associated with BVP (1.1)–(1.2)

$$\left\{ \left\langle \inf_{(u,y)\in\Xi_\Delta} J_\varepsilon(u,y) \right\rangle,\ \varepsilon \to 0 \right\}, \qquad (4.1)$$

where

$$\Xi_\Delta = \left\{ (u,y) \in L^p(\Omega) \times Y \ \middle|\ \begin{array}{c} \displaystyle\int_\Omega (\nabla y, \nabla\varphi)\,dx = \int_\Omega F'(y)\varphi\,dx \\ \displaystyle + \int_\Omega u\varphi\,dx, \quad \forall\,\varphi \in C_0^\infty(\Omega), \\ \displaystyle\int_\Omega |\nabla y|^2\,dx \leq \frac{2N}{N-2} \int_\Omega (F(y) - F(0))\,dx \\ \displaystyle -\frac{2}{N-2} \int_\Omega u\,(x - x_0, \nabla y)\,dx, \\ J_\varepsilon(u,y) < +\infty. \end{array} \right\}, \qquad (4.2)$$

$$J_\varepsilon(u,y) = \frac{\varepsilon}{2}\|\Delta y\|_{L^2(\Omega)}^2 + \frac{1}{p}\|g - u\|_{L^p(\Omega)}^p = \frac{\varepsilon}{2} \int_\Omega |\Delta y|^2\,dx + \frac{1}{p} \int_\Omega |g - u|^p\,dx. \qquad (4.3)$$

Here, $u \in L^p(\Omega)$ we consider as a fictitious control and ε is a small parameter. Hereinafter we assume that the parameter ε varies within a strictly decreasing sequence of positive real numbers which converge to zero.

It is worth to notice that, in contrast to the original BVP (1.1)–(1.2) for which it is unknown whether the set of its weak solutions is nonempty, each of the parametrized OCPs (4.1)–(4.3) is regular in the following sense: the set of feasible solution Ξ_Δ is always nonempty. Indeed, it is enough to take an arbitrary function $\widetilde{y} \in C_0^\infty(\Omega)$ and to put $\widetilde{u} := -\Delta\widetilde{y} - F'(\widetilde{y})$. Since $\widetilde{y} \in Y$ and $\Delta\widetilde{y} \in L^2(\Omega)$, it follows from Proposition 3.1 and Definition 2.1 that $J_\varepsilon(\widetilde{u}, \widetilde{y}) < +\infty$ and the pair $(\widetilde{u}, \widetilde{y})$ is related by integral identity (2.3) and inequality (3.1). Hence, $(\widetilde{u}, \widetilde{y}) \in \Xi_\Delta$.

Let us show that the OCPs (4.1)–(4.3) are solvable for each $\varepsilon > 0$.

Theorem 4.1. *Let $g \in L^p(\Omega)$ be a given distribution. Assume that Ω is star-shaped with respect to some of its interior point x_0. Then for every $\varepsilon > 0$ there exists at least one pair $\left(u_\varepsilon^0, y_\varepsilon^0\right) \in \Xi_\Delta$ such that*

$$J_\varepsilon\left(u_\varepsilon^0, y_\varepsilon^0\right) = \inf_{(u,y) \in \Xi_\Delta} J_\varepsilon(u, y).$$

Proof. Since $\Xi_\Delta \neq \emptyset$, it follows that for given $\varepsilon > 0$ and $g \in L^p(\Omega)$ there exists a minimizing sequence $\{(u_{\varepsilon,k}, y_{\varepsilon,k})\}_{k\in\mathbb{N}}$ to OCP (4.1)–(4.3), i.e.

$$\lim_{k\to\infty} J_\varepsilon(u_{\varepsilon,k}, y_{\varepsilon,k}) = \inf_{(u,y)\in\Xi_\varepsilon} J_\varepsilon(u, y) \leq J_\varepsilon(\widetilde{u}, \widetilde{y})$$

$$\leq \|\Delta\widetilde{y}\|_{L^2(\Omega)}^2 + \|g - \widetilde{u}\|_{L^p(\Omega)}^p = \widetilde{C} < +\infty. \qquad (4.4)$$

As a result, we have

$$\sup_{k\in N} \|\Delta y_{\varepsilon,k}\|_{L^2(\Omega)}^2 \overset{\text{by (4.4)}}{\leq} \varepsilon^{-1}\widetilde{C} \qquad (4.5)$$

and this estimate implies that each of the pairs $(u_{\varepsilon,k}, y_{\varepsilon,k})$ satisfies the inequality (see Proposition 3.1)

$$\left(\frac{N}{2} - 1\right) \int_\Omega |\nabla y_{\varepsilon,k}|^2 \, dx \leq N \int_\Omega (F(y_{\varepsilon,k}) - F(0)) \, dx$$

$$- \int_\Omega u_{\varepsilon,k}(x - x_0, \nabla y_{\varepsilon,k}) \, dx. \qquad (4.6)$$

Moreover, in view of Theorem 3.1, we have

$$\sup_{k\in N} \left[\|y_{\varepsilon,k}\|_{H_0^1(\Omega)}^2 + \|u_{\varepsilon,k}\|_{L^p(\Omega)}^p\right]$$

$$\overset{\text{by (4.5) and (3.7)}}{\leq} \sup_{k\in N} \left[\left(C_4\|u_{\varepsilon,k}\|_{L^p(\Omega)} + C_5\right)^2 + \|u_{\varepsilon,k}\|_{L^p(\Omega)}^p\right]$$

$$\leq 2^{p-1}C_5^p + p\left(1 + 2^{p-1}C_4^p\right) \sup_{k\in N} J_\varepsilon(u_{\varepsilon,k}, y_{\varepsilon,k}) \overset{\text{by (4.4)}}{<} +\infty \qquad (4.7)$$

for $\varepsilon > 0$ small enough, where the constants C_4 and C_5 do not depend on ε.

Thus, passing to subsequences, if necessary, we can suppose that there exists a pair $(u_\varepsilon^0, y_\varepsilon^0) \in L^p(\Omega) \times H_0^1(\Omega)$ such that

$$u_{\varepsilon,k} \rightharpoonup u_\varepsilon^0 \text{ in } L^p(\Omega) \text{ and } y_{\varepsilon,k} \rightharpoonup y_\varepsilon^0 \text{ in } H_0^1(\Omega) \text{ as } k \to \infty. \qquad (4.8)$$

Hence, in view of Proposition 3.2, y_ε^0 is a weak solution to BVP

$$-\Delta y = F'(y) + u_\varepsilon^0 \quad \text{in} \ \ \Omega,$$
$$y = 0 \quad \text{on} \ \ \partial\Omega,$$

the pair $(u_\varepsilon^0, y_\varepsilon^0)$ satisfies the inequality (4.6), and

$$F'(y_{\varepsilon,k}) \to F'(y_\varepsilon^0) \text{ in } L^1(\Omega) \text{ as } k \to \infty. \qquad (4.9)$$

Thus, $(u_\varepsilon^0, y_\varepsilon^0) \in \Xi$, where

$$\Xi = \left\{ (u,y) \in L^p(\Omega) \times Y \left|
\begin{array}{l}
\displaystyle\int_\Omega (\nabla y, \nabla \varphi) \, dx = \int_\Omega F'(y)\varphi \, dx \\
\displaystyle\quad + \int_\Omega u\varphi \, dx, \quad \forall \, \varphi \in C_0^\infty(\Omega), \\
\displaystyle\int_\Omega |\nabla y|^2 \, dx \le \frac{2N}{N-2} \int_\Omega (F(y) - F(0)) \, dx \\
\displaystyle\quad - \frac{2}{N-2} \int_\Omega u\,(x - x_0, \nabla y) \, dx.
\end{array}
\right.\right\}$$
$$(4.10)$$

It remains to show that $(u_\varepsilon^0, y_\varepsilon^0) \in \Xi_\Delta$ and $(u_\varepsilon^0, y_\varepsilon^0)$ is an optimal pair to the constrained minimization problem (4.1) for a given $\varepsilon > 0$. Taking into account the estimates (4.5) and (4.7), it is easy to see that the sequence $\{y_{\varepsilon,k}\}_{k\in\mathbb{N}}$ is bounded in the Banach space $H_{0,\Delta}^1(\Omega)$ connected with the Laplace operator by the formula (see, for instance, [9])

$$H_{0,\Delta}^1(\Omega) = \left\{ y \in H_0^1(\Omega) \ : \ \Delta y \in L^2(\Omega) \right\},$$

where the norm in $H_{0,\Delta}^1(\Omega)$ can be defined in the standard way as the norm of graph:

$$\|y\|_{H_{0,\Delta}^1(\Omega)}^2 = \|\nabla y\|_{L^2(\Omega)^N}^2 + \|\Delta y\|_{L^2(\Omega)}^2.$$

Hence, the limit properties (4.8) can be supplemented by the following one

$$\Delta y_{\varepsilon,k} \rightharpoonup \Delta y_\varepsilon^0 \text{ in } L^2(\Omega) \text{ as } k \to \infty.$$

As a result, making use of the lower semi-continuity property of the cost functional $J_\varepsilon : L^p(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ with respect to the weak convergence in $L^p(\Omega) \times H_{0,\Delta}^1(\Omega)$, we arrive at the following relation

$$\inf_{(u,y)\in\Xi_\Delta} J_\varepsilon(u,y) = \lim_{k\to\infty} J_\varepsilon(u_{\varepsilon,k}, y_{\varepsilon,k}) \ge J_\varepsilon(u_\varepsilon^0, y_\varepsilon^0).$$

Hence, $J_\varepsilon(u_\varepsilon^0, y_\varepsilon^0) < +\infty$ and $(u_\varepsilon^0, y_\varepsilon^0)$ is an optimal pair to the corresponding optimization problem (4.1). $\qquad\square$

Our next intention is to discuss the uniqueness property of the solutions to OCPs (4.1). We begin with the following noteworthy characteristic of the set Ξ_Δ.

Lemma 4.1. *Assume that, in addition to the property* (2.1)*, the function* $F \in C^1_{loc}(\mathbb{R})$ *is such that its derivative* $F' : \mathbb{R} \to (0, \infty)$ *is a strictly convex function. Then the set*

$$\Lambda = \{u \in L^p(\Omega) \ : \ \exists\, y \in Y \ \text{such that} \ (u, y) \in \Xi_\Delta\} \tag{4.11}$$

is nonempty and convex.

Proof. Since the set of feasible solution Ξ_Δ is nonempty, it follows from (4.11) that $\Lambda \neq \emptyset$. Let us establish the convexity of Λ. Let (u_1, y_1) and (u_2, y_2) be two different pairs of Ξ_Δ. It is clear that in this case we have $y_1 \neq y_2$ and $u_1, u_2 \in \Lambda$. Let $\lambda \in (0, 1)$. We set

$$u = \lambda u_1 + (1 - \lambda)u_2, \qquad y = \lambda y_1 + (1 - \lambda)y_2.$$

Our aim is to show that $u \in \Lambda$.

Since for Lipschitz domains Ω the mapping $-\Delta : H^1_0(\Omega) \to H^{-1}(\Omega)$ is an isomorphism (see Remark 2.2), we can define, in a unique way, a distribution $z \in H^1_0(\Omega)$ as follows

$$-\Delta z = u = \lambda u_1 + (1 - \lambda)u_2 \quad \text{in } \Omega.$$

Hence,

$$-\Delta y_1 = F'(y_1) + u_1 \quad \text{and} \quad -\Delta y_2 = F'(y_2) + u_2$$

imply

$$-\Delta y = \lambda F'(y_1) + (1 - \lambda)F'(y_2) + u \quad \text{in } \Omega. \tag{4.12}$$

Taking into account the facts that $F'(z) > 0$ almost everywhere in Ω (see (2.1)) and F' satisfies the Jensen's inequality

$$F'(y) \leq \lambda F'(y_1) + (1 - \lambda)F'(y_2), \tag{4.13}$$

we obtain

$$-\Delta z \leq F'(z) + u \quad \text{and} \quad \Delta y \overset{\text{by (4.12) and (4.13)}}{\geq} F'(y) + u, \tag{4.14}$$

i.e. z is a subsolution to the boundary value problem

$$-\Delta \psi = F'(\psi) + u \text{ in } \Omega, \qquad \psi = 0 \ \text{ on } \ \partial\Omega,$$

and y is its supersolution. Moreover, since

$$-\Delta y \geq F'(y) + u \overset{\text{by (2.1)}}{\geq} u = -\Delta z \text{ in } \Omega, \tag{4.15}$$

it follows that $-\Delta(y-z) \geq 0$. Hence, by the maximum principle we conclude that $y \geq z$ in Ω. Thus, following the classical techniques introduced by D.H. Sattinger [17] (see also H. Amann [1]), we deduce that for given control $u = \lambda u_1 + (1-\lambda)u_2$ there exists a solution ψ to the above boundary value problem such that

$$z(x) \leq \psi(x) \leq y(x) \quad \text{almost everywhere in } \Omega. \tag{4.16}$$

Moreover, as follows from (4.16) and the fact that $z, y \in H_0^1(\Omega)$, we have $\psi \in H_0^1(\Omega)$ and

$$\int_\Omega F'(\psi)\, dx \overset{\text{by (4.16) and (2.1)}}{\leq} \int_\Omega F'(y)\, dx$$

$$\overset{\text{by (4.13)}}{\leq} \lambda \|F'(y_1)\|_{L^1(\Omega)} + (1-\lambda)\|F'(y_2)\|_{L^1(\Omega)} < +\infty.$$

Hence, $\psi \in Y$ and, therefore, ψ is a weak solution to BVP (4.15).

In order to prove the inclusion $u \in \Lambda$, we have to establish that $(u, \psi) \in \Xi_\Delta$. To do so, we note that

$$-\Delta z = u \leq \underbrace{F'(\psi) + u}_{-\Delta\psi} \overset{\text{by (4.16)}}{\leq} F'(y) + u$$

$$\overset{\text{by (4.13)}}{\leq} \lambda\left(F'(y_1) + u_1\right) + (1-\lambda)\left(F'(y_2) + u_2\right)$$

$$= \lambda(-\Delta y_1) + (1-\lambda)(-\Delta y_2). \tag{4.17}$$

Since $\Delta z \in L^2(\Omega)$ and $y_i \in H_{0,\Delta}^1(\Omega)$, $i = 1, 2$, it follows from (4.17) that $\Delta\psi \in L^2(\Omega)$. Hence, $J_\varepsilon(u, \psi) < +\infty$ and it remains to note that ψ satisfies inequality (3.1) with $g = u$ by Proposition 3.1. So, (u, ψ) is an admissible pair to each of OCPs (4.1). Thus, $u \in \Lambda$ and, therefore, the convexity of Λ is established. $\square$

Remark 4.1. In general, we can not assert that the set Λ is closed in $L^p(\Omega)$. Indeed, let $\{u_k\}_{k \in \mathbb{N}} \subset \Lambda$ be a sequence such that $u_k \to u$ in $L^p(\Omega)$ as $k \to \infty$. Let $\{y_k\}_{k \in \mathbb{N}} \in Y$ be the corresponding sequence of states, that is, $(u_k, y_k) \in \Xi_\Delta$ for each $k \in \mathbb{N}$. Then, in view of estimate (3.7), we can suppose that there exists a distribution $y \in H_0^1(\Omega)$ such that $y_k \rightharpoonup y$ in $H_0^1(\Omega)$. Hence, by Proposition 3.2, we have: y is a weak solution to BVP (1.1)–(1.2) with $g = u$ and the pair (u, y) satisfies the inequality (3.1). However, it is unknown whether $J_\varepsilon(u, y) < +\infty$, because the given choice of the sequence of feasible pairs $\{(u_k, y_k)\}_{k \in \mathbb{N}} \subset \Xi_\Delta$ does not guarantee the $L^2(\Omega)$-boundedness of the corresponding Laplacians $\{\Delta y_k\}_{k \in \mathbb{N}}$. So, it is plausible to admit the case when $\Delta y \notin L^2(\Omega)$. As the same time, as follows from Theorem 4.1, the lack of $L^p(\Omega)$-closedness of Λ is not restrictive option for the solvability of the corresponding OCP. Moreover, as we show in the next assertion, the main result of Theorem 4.1 can be essentially specified.

Theorem 4.2. *Under assumptions of Lemma 4.1, each of the OCPs (4.1) has at most one solution.*

Proof. Let's assume the converse. Namely, let $\varepsilon > 0$ be a fixed value and let $(u_{\varepsilon,1}, y_{\varepsilon,1})$ and $(u_{\varepsilon,2}, y_{\varepsilon,2})$ be two different pairs such that $(u_{\varepsilon,1}, y_{\varepsilon,1}) \neq (u_{\varepsilon,2}, y_{\varepsilon,2})$,

$$(u_{\varepsilon,1}, y_{\varepsilon,1}), (u_{\varepsilon,2}, y_{\varepsilon,2}) \in \Xi_\Delta, \text{ and } J_\varepsilon(u_{\varepsilon,1}, y_{\varepsilon,1}) = J_\varepsilon(u_{\varepsilon,2}, y_{\varepsilon,2}) = \inf_{(u,y)\in\Xi_\Delta} J_\varepsilon(u, y).$$

We set $u_\varepsilon = (u_{\varepsilon,1} + u_{\varepsilon,2})/2$. By analogy with the proof of Lemma 4.1, it is easy to show that there exists a distribution $\psi_\varepsilon \in H_0^1(\Omega)$ such that $(u_\varepsilon, \psi_\varepsilon) \in \Xi_\Delta$ and

$$\psi_\varepsilon \leq (y_{\varepsilon,1} + y_{\varepsilon,2})/2 \quad \text{a.e. in } \Omega.$$

In fact, because of the strict convexity of F', it can be shown that the previous inequality is strict in Ω. Indeed, since

$$\begin{aligned}
-\Delta &\left(\frac{1}{2}y_{\varepsilon,1} + \frac{1}{2}y_{\varepsilon,2} - \psi_\varepsilon\right) \\
&= \frac{1}{2}\left(F'(y_{\varepsilon,1}) + u_{\varepsilon,1}\right) + \frac{1}{2}\left(F'(y_{\varepsilon,2}) + u_{\varepsilon,2}\right) - F'(\psi_\varepsilon) - u_\varepsilon \\
&= \frac{1}{2}\left(F'(y_{\varepsilon,1}) + F'(y_{\varepsilon,2})\right) - F'(\psi_\varepsilon) \\
&\geq \frac{1}{2}\left(F'(y_{\varepsilon,1}) + F'(y_{\varepsilon,2})\right) - F'\left(\frac{y_{\varepsilon,1} + y_{\varepsilon,2}}{2}\right) \geq 0
\end{aligned}$$

by the Jensen's inequality, and $\frac{1}{2}\left(F'(y_{\varepsilon,1}) + F'(y_{\varepsilon,2})\right) - F'\left(\frac{y_{\varepsilon,1}+y_{\varepsilon,2}}{2}\right) \neq 0$ on Ω because of the strict convexity of F', it follows that $(y_{\varepsilon,1} + y_{\varepsilon,2})/2 > \psi_\varepsilon$ in Ω by the strong maximum principle [13]. As a result, we obtain an inequality

$$\begin{aligned}
J_\varepsilon(u_\varepsilon, \psi_\varepsilon) &= \frac{\varepsilon}{2}\int_\Omega |\Delta\psi_\varepsilon|^2\,dx + \frac{1}{p}\int_\Omega |g - u_\varepsilon|^p\,dx \\
&\leq \frac{\varepsilon}{8}\int_\Omega |\Delta y_{\varepsilon,1} + \Delta y_{\varepsilon,2}|^2\,dx + \frac{1}{p\,2^p}\int_\Omega |g - u_{\varepsilon,1} + g - u_{\varepsilon,2}|^p\,dx \\
&< \frac{\varepsilon}{4}\int_\Omega |\Delta y_{\varepsilon,1}|^2\,dx + \frac{1}{2p}\int_\Omega |g - u_{\varepsilon,1}|^p\,dx + \frac{\varepsilon}{4}\int_\Omega |\Delta y_{\varepsilon,2}|^2\,dx + \frac{1}{2p}\int_\Omega |g - u_{\varepsilon,2}|^p\,dx \\
&= \frac{1}{2}J_\varepsilon(u_{\varepsilon,1}, y_{\varepsilon,1}) + \frac{1}{2}J_\varepsilon(u_{\varepsilon,2}, y_{\varepsilon,2}) = \inf_{(u,y)\in\Xi_\varepsilon} J_\varepsilon(u, y),
\end{aligned}$$

which is a contradiction with the fact that $(u_\varepsilon, \psi_\varepsilon)$ is an admissible pair to the problem (4.1). $\square$

Remark 4.2. As was mentioned in Remark 4.1, the convex set Λ is not closed in $L^p(\Omega)$. Let $\overline{\Lambda} \subset L^p(\Omega)$ denotes the closure of Λ with respect to the strong topology of $L^p(\Omega)$. Then by Mazur's theorem this set coincides with the sequential weak closure of Λ in $L^p(\Omega)$, that is, $\overline{\Lambda} = \text{cl}_{w-L^p(\Omega)}\Lambda$.

5. Variational Properties of Fictitious Optimal Control Problems

Before setting foot in the asymptotic analysis of the sequence of OCPs (4.1) as $\varepsilon \to 0$, we define the μ-topology on $L^p(\Omega) \times H^1(\Omega)$ as the product of weak

topologies of $L^p(\Omega)$ and $H^1(\Omega)$, respectively. Let $\mathrm{cl}_\mu \Xi_\Delta$ be the sequential closure of the set of feasible pairs Ξ_Δ with respect to the μ-topology. In view of Proposition 3.2, it is clear that in this case we have

$$\overline{\Lambda} = \{ u \in L^p(\Omega) \ : \ \exists\, y \in Y \ \text{ such that } \ (u, y) \in \mathrm{cl}_\mu \Xi_\Delta \}. \tag{5.1}$$

Indeed, let u be an arbitrary element of $\overline{\Lambda}$. Then there exists a sequence $\{u_\varepsilon\}_{\varepsilon>0} \in \Lambda$ such that $u_\varepsilon \to u$ in $L^p(\Omega)$ as $\varepsilon \to 0$. By definition of the set Λ, it follows that we can construct a sequence of pairs $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon>0}$ such that $(u_\varepsilon, y_\varepsilon) \in \Xi_\Delta$ for all $\varepsilon > 0$. In view of definition of the set Ξ_Δ and Theorem 3.1, the sequence of states $\{y_\varepsilon\}_{\varepsilon>0}$ is bounded in $H_0^1(\Omega)$. So, we can suppose that the exists an element $y \in H_0^1(\Omega)$ such that, up to a subsequence, $y_\varepsilon \rightharpoonup y$ in $H_0^1(\Omega)$. Hence, $(u_\varepsilon, y_\varepsilon) \overset{\mu}{\longrightarrow} (u, y)$ in $L^p(\Omega) \times H^1(\Omega)$ and therefore $(u, y) \in \mathrm{cl}_\mu \Xi_\Delta$. As a result, we obtain $u \in \{ v \in L^p(\Omega) \ : \ \exists\, y \in Y \ \text{ such that } \ (v, y) \in \mathrm{cl}_\mu \Xi_\Delta \}$, that is, we have shown that

$$\overline{\Lambda} \subseteq \{ u \in L^p(\Omega) \ : \ \exists\, y \in Y \ \text{ such that } \ (u, y) \in \mathrm{cl}_\mu \Xi_\Delta \}. \tag{5.2}$$

In order to establish the converse inclusion, we fix an arbitrary pair (u^*, y^*) in $\mathrm{cl}_\mu \Xi_\Delta$. Then $u^* \in \{ u \in L^p(\Omega) \ : \ \exists\, y \in Y \ \text{ such that } \ (u, y) \in \mathrm{cl}_\mu \Xi_\Delta \}$ and there exists a sequence $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon>0} \subset L^p(\Omega) \times Y$ such that $(u_\varepsilon, y_\varepsilon) \in \Xi_\Delta$ for each $\varepsilon > 0$ and $(u_\varepsilon, y_\varepsilon) \overset{\mu}{\longrightarrow} (u^*, y^*)$ in $L^p(\Omega) \times H^1(\Omega)$. Since the condition $(u_\varepsilon, y_\varepsilon) \in \Xi_\Delta$ implies that $u_\varepsilon \in \Lambda$, it follows that $\{u_\varepsilon\}_{\varepsilon>0} \subset \Lambda$ and $u_\varepsilon \rightharpoonup u^*$ in $L^p(\Omega)$ as $\varepsilon \to 0$. Hence, $u^* \in \mathrm{cl}_{w-L^p(\Omega)} \Lambda$. To conclude, it is enough to apply the Mazur's Theorem. Thus,

$$\{ u \in L^p(\Omega) \ : \ \exists\, y \in Y \ \text{ such that } \ (u, y) \in \mathrm{cl}_\mu \Xi_\Delta \} \subseteq \overline{\Lambda}$$

and combining this fact with (5.2), we arrive at the required equality.

It is clear now that $\mathrm{cl}_\mu \Xi_\Delta \subseteq \Xi$, where the set Ξ is defined in (4.10). However, we can not exclude the case when we have $(u, y) \in \Xi$ and $u \in L^p(\Omega) \setminus \overline{\Lambda}$. Hence, the validity of the inclusion $\Xi \subseteq \mathrm{cl}_\mu \Xi_\Delta$ is an open question. So, our next intention is to specify the structure of the set $\mathrm{cl}_\mu \Xi_\Delta$.

Lemma 5.1. *Assume that Ω is star-shaped with respect to some of its interior point x_0. Assume also that, in addition to the property (2.1), the function $F \in C_{loc}^1(\mathbb{R})$ is such that its derivative $F' : \mathbb{R} \to (0, \infty)$ is a strictly convex function. Then*

$$\mathrm{cl}_\mu \Xi_\Delta = \left\{ (u, y) \in \overline{\Lambda} \times Y \ \left| \ \begin{array}{l} \displaystyle\int_\Omega (\nabla y, \nabla \varphi)\, dx = \int_\Omega F'(y)\varphi\, dx \\ \displaystyle \qquad + \int_\Omega u\varphi\, dx, \quad \forall\, \varphi \in C_0^\infty(\Omega), \\ \displaystyle\int_\Omega |\nabla y|^2\, dx \le \frac{2N}{N-2} \int_\Omega (F(y) - F(0))\, dx \\ \displaystyle \qquad - \frac{2}{N-2} \int_\Omega u\,(x - x_0, \nabla y)\, dx. \end{array} \right. \right\} \tag{5.3}$$

Proof. Since the set in the right-hand side of (5.3) can be represented as

$$\Xi \cap \left(\overline{\Lambda} \times Y \right)$$

and the inclusion $\mathrm{cl}_\mu \, \Xi_\Delta \subseteq \Xi \cap \left(\overline{\Lambda} \times Y \right)$ is obvious, we concentrate at the proof of the reverse inclusion

$$\Xi \cap \left(\overline{\Lambda} \times Y \right) \subseteq \mathrm{cl}_\mu \, \Xi_\Delta. \tag{5.4}$$

Let (u^*, y^*) be an arbitrary representative of the set $\Xi \cap \left(\overline{\Lambda} \times Y \right)$. Then y^* is a weak solution to the boundary value problem (1.1)–(1.2) with $g = u^*$ in the sense of Definition 2.1. Our aim is to show that $(u^*, y^*) \in \mathrm{cl}_\mu \, \Xi_\Delta$. To this end, it is enough to prove the existence of the sequence $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon > 0}$ in $L^p(\Omega) \times H_0^1(\Omega)$ such that

$$(u_\varepsilon, y_\varepsilon) \overset{\mu}{\longrightarrow} (u^*, y^*) \quad \text{in} \quad L^p(\Omega) \times H_0^1(\Omega), \quad \text{and} \quad (u_\varepsilon, y_\varepsilon) \in \Xi_\Delta \; \forall \, \varepsilon > 0. \tag{5.5}$$

The most natural way to construct such sequences is to apply the procedure of the direct smoothing. Indeed, let us define the elements $y_\varepsilon \in H_0^1(\Omega)$ as follows

$$y_\varepsilon(x) = \frac{1}{\nu^N(\varepsilon)} \int_{\mathbb{R}^N} K\left(\frac{x - z}{\nu(\varepsilon)} \right) \widetilde{\varphi_\varepsilon}(z) \, dz, \tag{5.6}$$

where $\nu(\varepsilon) > 0$ is a positive value such that $\nu(\varepsilon) \to 0$ as $\varepsilon \to 0$, K is a positive compactly supported smooth function with properties

$$K \in C_0^\infty(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} K(x) \, dx = 1, \quad \text{and} \quad K(x) = K(-x), \tag{5.7}$$

$\widetilde{\ }$ is zero extension operator outside of Ω, and $\{\varphi_\varepsilon\}_{\varepsilon > 0}$ is a sequence in $C_0^\infty(\Omega)$ such that $\varphi_\varepsilon \to y^*$ in $H_0^1(\Omega)$ as $\varepsilon \to 0$.

Then the property

$$y_\varepsilon \to y^* \quad \text{in} \quad L^2(\Omega) \quad \text{and} \quad \nabla y_\varepsilon \to \nabla y^* \quad \text{in} \quad L^2(\Omega)^N \tag{5.8}$$

is the direct consequence of the classical properties of smoothing. Moreover, since each element of the sequence $\{\varphi_\varepsilon\}_{\varepsilon > 0}$ has a compact support in Ω, we can suppose that y_ε has zero trace on $\partial\Omega$ for each ε small enough, i.e., in view of (5.8) we have: $y_\varepsilon \in H_0^1(\Omega)$ for each $\varepsilon > 0$. It remains to note that the parameters $\nu(\varepsilon)$ can be defined such that $\lim_{\varepsilon \to 0} \sqrt{\varepsilon}/\nu^2(\varepsilon) = 0$. Hence,

$$\sqrt{\varepsilon} \Delta y_\varepsilon = \frac{\sqrt{\varepsilon}}{\nu^2(\varepsilon)} \left[\frac{1}{\nu^N(\varepsilon)} \int_{\mathbb{R}^N} \Delta K\left(\frac{x - z}{\nu(\varepsilon)} \right) \widetilde{\varphi_\varepsilon}(z) \, dz \right] \to 0 \quad \text{in} \quad L^2(\Omega).$$

Summarizing these properties, we can infer that for a given element $y^* \in Y \subset H_0^1(\Omega)$ there exists a sequence $\{y_\varepsilon\}_{\varepsilon > 0}$ such that

(a) $y_\varepsilon \in Y$ for each $\varepsilon > 0$ and $y_\varepsilon \to y^*$ in $H_0^1(\Omega)$ as $\varepsilon \to 0$;

(b) $\Delta y_\varepsilon \in L^2(\Omega)$ for each $\varepsilon > 0$ and $\lim_{\varepsilon \to 0} \left(\varepsilon \|\Delta y_\varepsilon\|_{L^2(\Omega)}^2 \right) = 0$.

Moreover, by Lemma 2.1, we have $F'(y^*) \in H^{-1}(\Omega)$. Hence, we can supplement the above properties (a)–(b) by the following one

(c) $\sup_{\varepsilon>0} \|F'(y_\varepsilon)\|_{H^{-1}(\Omega)} < +\infty$.

Remark 5.1. As immediately follows from the reasons given above, the limit property (b) for elements $y_\varepsilon \in H_0^1(\Omega)$ can be sharpened in the following way: the smoothing parameter $\nu(\varepsilon)$ can be defined such that for a given non-negative constant C_0 we have

(b)* $\Delta y_\varepsilon \in L^2(\Omega)$ for each $\varepsilon > 0$ and $\lim_{\varepsilon \to 0} \left(\varepsilon \|\Delta y_\varepsilon\|^2_{L^2(\Omega)} \right) = C_0$.

Let us define the corresponding controls $\{u_\varepsilon\}_{\varepsilon>0}$ as follows

$$u_\varepsilon := -\Delta y_\varepsilon - F'(y_\varepsilon), \quad \forall \varepsilon > 0.$$

Since y_ε are the smooth functions, we obviously have $u_\varepsilon \in L^p(\Omega)$. Moreover, by Proposition 3.1, each of the pairs $(u_\varepsilon, y_\varepsilon) \in L^p(\Omega) \times Y$ is related by the integral identity

$$\int_\Omega (\nabla y_\varepsilon, \nabla \varphi) \, dx = \int_\Omega F'(y_\varepsilon) \varphi \, dx + \int_\Omega u_\varepsilon \varphi \, dx, \quad \forall \varphi \in C_0^\infty(\Omega)$$

and the inequality

$$\left(\frac{N}{2} - 1 \right) \int_\Omega |\nabla y_\varepsilon|^2 \, dx \le N \int_\Omega (F(y_\varepsilon) - F(0)) \, dx - \int_\Omega u_\varepsilon (x - x_0, \nabla y_\varepsilon) \, dx.$$

Thus, the sequence $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon>0}$ lies in the set Ξ_Δ. It remains to show that $(u_\varepsilon, y_\varepsilon) \xrightarrow{\mu} (u^*, y^*)$ in $L^p(\Omega) \times H_0^1(\Omega)$. With that in mind, we make use of the energy equality

$$\int_\Omega |\nabla y_\varepsilon|^2 \, dx = \int_\Omega y_\varepsilon F'(y_\varepsilon) \, dx + \int_\Omega u_\varepsilon y_\varepsilon \, dx, \quad \forall \varepsilon > 0. \tag{5.9}$$

Since

$$\int_\Omega y_\varepsilon F'(y_\varepsilon) \, dx \le \|F'(y_\varepsilon)\|_{H^{-1}(\Omega)} \|y_\varepsilon\|_{H_0^1(\Omega)},$$

and

$$\|y_\varepsilon\|_{H_0^1(\Omega)} = \left(\int_\Omega |\nabla y_\varepsilon|^2 \, dx \right)^{1/2} \le C_s \|y\|_{L^q(\Omega)}$$

by the Sobolev Embedding Theorem for $q \in \left[1, \frac{2N}{N-2}\right]$, it follows from (5.9) that

$$\left| \int_\Omega u_\varepsilon y_\varepsilon \, dx \right| \le \|y_\varepsilon\|^2_{H_0^1(\Omega)} + \|F'(y_\varepsilon)\|_{H^{-1}(\Omega)} \|y_\varepsilon\|_{H_0^1(\Omega)}$$

$$\le C_s \left(\|y_\varepsilon\|_{H_0^1(\Omega)} + \|F'(y_\varepsilon)\|_{H^{-1}(\Omega)} \right) \|y\|_{L^{p'}(\Omega)},$$

where $p' = p/(p-1) \in \left[1, \frac{2N}{N-2}\right]$ for a given $p > 2$. Hence,

$$\sup_{\varepsilon>0} \|u_\varepsilon\|_{L^p(\Omega)} \le C_s \left(\sup_{\varepsilon>0} \|y_\varepsilon\|_{H_0^1(\Omega)} + \sup_{\varepsilon>0} \|F'(y_\varepsilon)\|_{H^{-1}(\Omega)}\right) < +\infty.$$

As a result, we deduce that the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ is relatively compact with respect to the weak topology of $L^p(\Omega)$. Let $u^0 \in L^p(\Omega)$ be any its cluster point and $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ be a subsequence of $\{u_\varepsilon\}_{\varepsilon>0}$ such that $u_{\varepsilon_k} \rightharpoonup u^0$ in $L^p(\Omega)$ as $k \to \infty$. Since $\{(u_{\varepsilon_k}, y_{\varepsilon_k})\}_{k\in\mathbb{N}} \subset \Xi_\Delta$, it follows that each of the pairs $(u_{\varepsilon_k}, y_{\varepsilon_k})$ is related by the integral identity

$$\int_\Omega (\nabla y_{\varepsilon_k}, \nabla\varphi)\, dx = \int_\Omega F'(y_{\varepsilon_k})\varphi\, dx + \int_\Omega u_{\varepsilon_k}\varphi\, dx, \quad \forall\varphi \in C_0^\infty(\Omega). \qquad (5.10)$$

Taking into account that $y_{\varepsilon_k} \to y^*$ in $H_0^1(\Omega)$, we can pass to the limit in (5.10) as $k \to \infty$ (see the proof of Proposition 3.2 for the details). We get

$$\int_\Omega (\nabla y^*, \nabla\varphi)\, dx = \int_\Omega F'(y^*)\varphi\, dx + \int_\Omega u^0\varphi\, dx, \quad \forall\varphi \in C_0^\infty(\Omega). \qquad (5.11)$$

On the other hand, $(u^*, y^*) \in \Xi \cap (\overline{\Lambda} \times Y)$. Hence, this pair is related by the similar relation

$$\int_\Omega (\nabla y^*, \nabla\varphi)\, dx = \int_\Omega F'(y^*)\varphi\, dx + \int_\Omega u^*\varphi\, dx, \quad \forall\varphi \in C_0^\infty(\Omega). \qquad (5.12)$$

Combining (5.11) with (5.12), we obtain

$$\int_\Omega \left(u^0 - u^*\right)\varphi\, dx = 0, \quad \forall\varphi \in C_0^\infty(\Omega).$$

Since $C_0^\infty(\Omega)$ is dense in $L^{p'}(\Omega)$, it follows that $u^0 = u^*$ almost everywhere in Ω. It remains to note that this inference is valid for any cluster point u^0 of the sequence $\{u_\varepsilon\}_{\varepsilon>0}$. Hence, u^* is a weak limit in $L^p(\Omega)$ for the entire sequence $\{u_\varepsilon\}_{\varepsilon>0}$.

Thus, we have constructed a sequence $\{(u_\varepsilon, y_\varepsilon)\}_{\varepsilon>0}$ in $L^p(\Omega) \times H_0^1(\Omega)$ satisfying the properties (5.5). It suffices to conclude that $(u^*, y^*) \in \mathrm{cl}_\mu\, \Xi_\Delta$. Hence, the inclusion (5.4) is valid. The proof is complete. $\qquad\square$

As a consequence of this lemma, we make use of the following observation.

Proposition 5.1. Under assumptions of Lemma 5.1, we have: for any $(u^*, y^*) \in \mathrm{cl}_\mu\, \Xi_\Delta$ and for any non-negative real number C_0 there exists a sequence of pairs $\{(\widehat{u}_\varepsilon, \widehat{y}_\varepsilon)\}_{\varepsilon>0}$ in $L^p(\Omega) \times H_0^1(\Omega)$ such that

$$(\widehat{u}_\varepsilon, \widehat{y}_\varepsilon) \overset{\mu}{\rightharpoonup} (u^*, \widehat{y})\ \text{ in }\ L^p(\Omega) \times H_0^1(\Omega), \quad (\widehat{u}_\varepsilon, \widehat{y}_\varepsilon) \in \Xi_\Delta, \quad \forall\varepsilon > 0, \qquad (5.13)$$

$$\widehat{u}_\varepsilon \to u^*\ \text{strongly in}\ L^p(\Omega), \qquad (5.14)$$

$$\Delta\widehat{y}_\varepsilon \in L^2(\Omega) \quad \forall\varepsilon > 0, \quad \text{and} \quad \lim_{\varepsilon\to 0}\left(\frac{\varepsilon}{2}\|\Delta\widehat{y}_\varepsilon\|_{L^2(\Omega)}^2\right) = C_0, \qquad (5.15)$$

where $\widehat{y} = y^*$ provided the Dirichlet boundary value problem (1.1)–(1.2) has a unique solution for $g = u^*$.

Proof. Let $(u^*, y^*) \in \mathrm{cl}_\mu \, \Xi_\Delta$ be an arbitrary pair. Closely following the proof of Lemma 5.1, we can construct a sequence $\{(u_{\varepsilon_k}, y_{\varepsilon_k})\}_{k \in \mathbb{N}}$ in $L^p(\Omega) \times H_0^1(\Omega)$ with properties (5.5) and (a)–(c). Here, $\{\varepsilon_k\}_{k \in \mathbb{N}}$ stands for a strictly decreasing sequence of positive real numbers converging to zero. Then, by Mazur's lemma, there exists a sequence of controls $\{\widehat{u}_{\varepsilon_k}\}_{k \in N}$ such that

$$\widehat{u}_{\varepsilon_k} \to u^* \text{ strongly in } L^p(\Omega) \text{ and } \widehat{u}_{\varepsilon_k} = \mathrm{co}\,\{u_{\varepsilon_1}, \dots, u_{\varepsilon_k}\}, \ \forall\, k \in \mathbb{N}, \qquad (5.16)$$

where $\mathrm{co}\, A$ denotes the convex hull of the set A. In view of Lemma 4.1, Λ is a convex set. Since $\{u_{\varepsilon_k}\}_{k \in \mathbb{N}} \subset \Lambda$, it follows from (5.16) that $\{\widehat{u}_{\varepsilon_k}\}_{k \in \mathbb{N}} \subset \Lambda$ as well. Let us show that the corresponding weak solutions $\widehat{y}_k$ to the boundary value problem

$$-\Delta y = F'(y) + \widehat{u}_{\varepsilon_k} \quad \text{in } \Omega, \qquad y = 0 \quad \text{on } \partial\Omega,$$

satisfy properties (5.13) and (5.15).

By definition of the convex hull $\mathrm{co}\,\{u_{\varepsilon_1}, u_{\varepsilon_2}, \dots, u_{\varepsilon_k}\}$ we have: for each $k \in \mathbb{N}$ there exists a collection of non-negative real numbers $\{\alpha_{1,k}, \alpha_{2,k}, \dots, \alpha_{k,k}\}$ such that

$$\alpha_{1,k} + \alpha_{2,k} + \dots + \alpha_{k,k} = 1 \quad \text{and} \quad \widehat{u}_\varepsilon = \alpha_{1,k} u_{\varepsilon_1} + \alpha_{2,k} u_{\varepsilon_2} + \dots + \alpha_{k,k} u_{\varepsilon_k}.$$

With each control $\widehat{u}_{\varepsilon_k}$ we associate two elements $z_k \in H_0^1(\Omega)$ and $w_k \in H_0^1(\Omega)$ by the following rule

$$-\Delta z_k = \widehat{u}_{\varepsilon_k} = \sum_{i=1}^k \alpha_{i,k} u_{\varepsilon_i} \quad \text{in } \Omega, \qquad\qquad (5.17)$$

$$w_k = \alpha_{1,k} y_{\varepsilon_1} + \alpha_{2,k} y_{\varepsilon_2} + \dots + \alpha_{k,k} y_{\varepsilon_k}. \qquad\qquad (5.18)$$

Hence,

$$-\Delta y_{\varepsilon_i} = F'(y_{\varepsilon_i}) + u_{\varepsilon_i}, \quad i = 1, \dots, k$$

imply

$$-\Delta w_k = \sum_{i=1}^k \alpha_{i,k} F'(y_{i,k}) + \widehat{u}_{\varepsilon_k} \quad \text{in } \Omega. \qquad\qquad (5.19)$$

Taking into account the facts that $F'(y) > 0$ almost everywhere in Ω (see (2.1)) and F' satisfies Jensen's inequality

$$F'(w_k) \le \sum_{i=1}^k \alpha_{i,k} F'(y_{i,k}), \qquad\qquad (5.20)$$

we obtain

$$-\Delta z_k \le F'(z_k) + \widehat{u}_{\varepsilon_k} \quad \text{and} \quad \Delta w_k \overset{\text{by (5.19) and (5.20)}}{\ge} F'(w_k) + \widehat{u}_{\varepsilon_k}, \qquad (5.21)$$

i.e. z_k is a subsolution to the boundary value problem

$$-\Delta\psi = F'(\psi) + \widehat{u}_{\varepsilon_k} \text{ in } \Omega, \qquad \psi = 0 \text{ on } \partial\Omega,$$

and w_k is its supersolution. Moreover, since

$$-\Delta w_k \geq F'(w_k) + \widehat{u}_{\varepsilon_k} \overset{\text{by (2.1)}}{\geq} \widehat{u}_{\varepsilon_k} = -\Delta z_k \text{ in } \Omega, \qquad (5.22)$$

it follows that $-\Delta(w_k - z_k) \geq 0$. Hence, by the maximum principle we conclude that $w_k \geq z_k$ in Ω. Thus, following the classical techniques, we deduce that for given control $\widehat{u}_{\varepsilon_k}$ there exists a solution $\widehat{y}_{\varepsilon_k}$ to the above boundary value problem such that

$$z_k(x) \leq \widehat{y}_{\varepsilon_k}(x) \leq w_k(x) \quad \text{almost everywhere in } \Omega. \qquad (5.23)$$

Moreover, as follows from (5.23) and the fact that $z_k, w_k \in H_0^1(\Omega)$, we have $\widehat{y}_{\varepsilon_k} \in H_0^1(\Omega)$ and

$$\int_\Omega F'(\widehat{y}_{\varepsilon_k}) \, dx \overset{\text{by (5.23) and (2.1)}}{\leq} \int_\Omega F'(\omega_k) \, dx$$

$$\overset{\text{by (5.20)}}{\leq} \sum_{i=1}^k \alpha_{i,k} \|F'(y_{i,k})\|_{L^1(\Omega)} < +\infty.$$

Hence, $\widehat{y}_{\varepsilon_k} \in Y$ and, therefore, $\widehat{y}_{\varepsilon_k}$ is a weak solution to BVP (5.22).

Let us show that $\{\widehat{y}_{\varepsilon_k}\}_{k\in\mathbb{N}}$ is a weakly compact sequence in $H_0^1(\Omega)$ with the extra property

$$\Delta\widehat{y}_{\varepsilon_k} \in L^2(\Omega) \quad \forall\varepsilon > 0, \quad \text{and} \quad \lim_{\varepsilon_k\to 0}\left(\frac{\varepsilon_k}{2}\|\Delta\widehat{y}_{\varepsilon_k}\|_{L^2(\Omega)}^2\right) = C_0. \qquad (5.24)$$

Indeed, as follows from (5.17), we have

$$\sup_{k\in\mathbb{N}}\|z_k\|_{H_0^1(\Omega)} = \sup_{k\in\mathbb{N}}\|(-\Delta)^{-1}\widehat{u}_{\varepsilon_k}\|_{H_0^1(\Omega)} \leq C_\Omega \sup_{k\in\mathbb{N}}\left(\sum_{i=1}^k \alpha_{i,k}\|u_{\varepsilon_i}\|_{L^2(\Omega)}\right)$$

$$\leq C_\Omega|\Omega|^{\frac{p-2}{2p}}\sup_{k\in\mathbb{N}}\left(\sum_{i=1}^k \alpha_{i,k}\|u_{\varepsilon_i}\|_{L^p(\Omega)}\right)$$

$$\leq C_\Omega|\Omega|^{\frac{p-2}{2p}}\sup_{k\in\mathbb{N}}\|u_{\varepsilon_k}\|_{L^p(\Omega)} < +\infty \qquad (5.25)$$

and

$$\sup_{k\in\mathbb{N}}\|\Delta z_k\|_{L^2(\Omega)} = \sup_{k\in\mathbb{N}}\|\widehat{u}_{\varepsilon_k}\|_{L^2(\Omega)} \leq \sup_{k\in\mathbb{N}}\left[\left(\sum_{i=1}^k \alpha_{i,k}\right)\|u_{\varepsilon_k}\|_{L^2(\Omega)}\right]$$

$$\leq |\Omega|^{\frac{p-2}{2p}}\sup_{k\in\mathbb{N}}\|u_{\varepsilon_k}\|_{L^p(\Omega)} < +\infty, \qquad (5.26)$$

where the constant C_Ω comes from the Friedrichs–Poincaré inequality.

As for the super solutions w_k, we get

$$
\sup_{k \in \mathbb{N}} \| w_k \|_{H_0^1(\Omega)} = \sup_{k \in \mathbb{N}} \| \sum_{i=1}^{k} \alpha_{i,k} y_{\varepsilon_i} \|_{H_0^1(\Omega)}
$$

$$
\text{(by (3.7) and the fact that } (u_{\varepsilon_k}, y_{\varepsilon_k}) \in \Xi_\Delta)
$$

$$
\leq \sup_{k \in \mathbb{N}} \left[\left(\sum_{i=1}^{k} \alpha_{i,k} \right) \left(C_4 \| u_{\varepsilon_k} \|_{L^p(\Omega)} + C_5 \right) \right]
$$

$$
= C_4 \sup_{k \in \mathbb{N}} \| u_{\varepsilon_k} \|_{L^p(\Omega)} + C_5 < +\infty \qquad (5.27)
$$

and (see Remark 5.1 for the details)

$$
\lim_{\varepsilon_k \to 0} \left(\sqrt{\varepsilon_k} \| \Delta w_k \|_{L^2(\Omega)} \right) \leq \lim_{\varepsilon_k \to 0} \left(\sqrt{\varepsilon_k} \left[\sum_{i=1}^{k} \alpha_{i,k} \| \Delta y_{\varepsilon_i} \|_{L^2(\Omega)} \right] \right)
$$

$$
\leq \lim_{\varepsilon_k \to 0} \left[\sqrt{\varepsilon_k} \| \Delta y_{\varepsilon_k} \|_{L^2(\Omega)} \right] \overset{\text{by property } (b)^*}{=} 0. \qquad (5.28)
$$

As a result, it follows from inequality (5.23) and estimates (5.25), and (5.27) that the sequence $\{ \widehat{y}_{\varepsilon_k} \}_{k \in \mathbb{N}}$ is bounded in $H_0^1(\Omega)$ and, hence, there exists an element $\widehat{y} \in H_0^1(\Omega)$ such that, up to a subsequence,

$$
\widehat{y}_{\varepsilon_k} \rightharpoonup \widehat{y} \quad \text{in} \quad H_0^1(\Omega).
$$

Then, Proposition 3.2 implies that $\widehat{y}$ is a weak solution to BVP (1.1)–(1.2) for $g = u^*$ and the pair $(u^*, \widehat{y})$ satisfies the inequality (3.1). It is clear now that $\widehat{y} = y^*$ provided the Dirichlet boundary value problem (1.1)–(1.2) has a unique solution for $g = u^*$. Thus, the sequence $\{ (\widehat{u}_\varepsilon, \widehat{y}_\varepsilon) \}_{\varepsilon > 0}$ possesses desired properties (5.13)–(5.14). As for the property (5.15), its validity immediately follows from (5.26) and (5.28). The proof is complete. $\qquad \square$

Our final intention in this paper is to discuss the variational properties of the solutions to the sequence of fictitious optimal control problems (4.1) as $\varepsilon \to 0$. As usual, we assume that Ω is star-shaped with respect to some of its interior point x_0 and in addition to the property (2.1), the function $F' : \mathbb{R} \to (0, \infty)$ is strictly convex. Let $\left\{ (u_\varepsilon^0, y_\varepsilon^0) \right\}_{\varepsilon > 0} \subset \Xi_\Delta$ be a sequence of optimal pairs to the corresponding fictitious problem (4.1). As follows from Theorem 4.2, each of the OCPs (4.1) has a unique solution. We begin with the following result.

Proposition 5.2. There exists a pair $(u^0, y^0) \in \mathrm{cl}_\mu \, \Xi_\Delta$ such that, within a subsequence,

$$
(u_\varepsilon^0, y_\varepsilon^0) \overset{\mu}{\rightharpoonup} (u^0, y^0) \quad \text{in} \quad L^p(\Omega) \times H_0^1(\Omega), \qquad (5.29)
$$

$$
\lim_{\varepsilon \to 0} \left(\frac{\varepsilon}{2} \| \Delta y_\varepsilon^0 \|_{L^2(\Omega)}^2 \right) = C_\Delta \qquad (5.30)
$$

for some $C_\Delta \geq 0$.

Proof. Since the pair $(\widetilde{u}, \widetilde{y}) := (-F'(0), 0)$ is feasible for each of OCPs (4.1), i.e. $(\widetilde{u}, \widetilde{y}) \in \Xi_\Delta$, it follows that

$$J_\varepsilon(u_\varepsilon^0, y_\varepsilon^0) = \inf_{(u,y)\in\Xi_\Delta} J_\varepsilon(u, y) \leq J_\varepsilon(\widetilde{u}, \widetilde{y}) = \underbrace{\frac{1}{p}\|g + F'(0)\|_{L^p(\Omega)}}_{C^*} < +\infty, \quad \forall \varepsilon > 0.$$

Hence,

$$\sup_{\varepsilon>0}\left[\frac{\varepsilon}{2}\|\Delta y_\varepsilon^0\|_{L^2(\Omega)}^2\right] \leq C^* \quad \text{and}$$

$$\sup_{\varepsilon>0}\|u_\varepsilon^0\|_{L^p(\Omega)}^p \leq 2^{p-1}\|g\|_{L^p(\Omega)}^p + 2^{p-1}\sup_{\varepsilon>0}\|g - u_\varepsilon^0\|_{L^p(\Omega)}^p$$

$$\leq 2^{p-1}\|g\|_{L^p(\Omega)}^p + 2^{p-1}p\sup_{\varepsilon>0} J_\varepsilon(u_\varepsilon^0, y_\varepsilon^0) \leq 2^{p-1}\|g\|_{L^p(\Omega)}^p + 2^{p-1}pC^*,$$

$$\sup_{\varepsilon>0}\|y_\varepsilon^0\|_{H_0^1(\Omega)} \overset{\text{by (3.7)}}{\leq} C_4\sup_{\varepsilon>0}\|u_\varepsilon^0\|_{L^p(\Omega)} + C_5 < +\infty$$

So, we can suppose that there exists a subsequence of $\left\{(u_\varepsilon^0, y_\varepsilon^0)\right\}_{\varepsilon>0}$ (still denoted by the same index ε), a pair $(u^0, y^0) \in L^p(\Omega) \times H_0^1(\Omega)$, and a constant $C_\Delta \geq 0$ such that

$$\lim_{\varepsilon\to 0}\left[\frac{\varepsilon}{2}\|\Delta y_\varepsilon^0\|_{L^2(\Omega)}^2\right] = C_\Delta \quad \text{and} \quad (u_\varepsilon^0, y_\varepsilon^0) \overset{\mu}{\to} (u^0, y^0) \text{ in } L^p(\Omega) \times H_0^1(\Omega). \quad (5.31)$$

To conclude the proof, it remains to note that $(u^0, y^0) \in \mathrm{cl}_\mu\, \Xi_\Delta$ by Proposition 3.2 and Lemma 5.1. $\qquad\square$

The main question arising in this case is about variational properties of the μ-cluster pairs $(u^0, y^0) \in L^p(\Omega) \times H_0^1(\Omega)$.

Theorem 5.1. *Let $(u^0, y^0) \in L^p(\Omega) \times H_0^1(\Omega)$ be a μ-cluster pair of the sequence of optimal solutions $\left\{(u_\varepsilon^0, y_\varepsilon^0)\right\}_{\varepsilon>0}$ to the fictitious problems (4.1) as $\varepsilon \to 0$. Then*

$$J_0(u^0, y^0) = \inf_{(u,y)\in\mathrm{cl}_\mu\, \Xi_\Delta} J_0(u, y), \quad\quad\quad (5.32)$$

where

$$J_0(u, y) = \frac{1}{p}\|g - u\|_{L^p(\Omega)}^p, \quad \forall (u, y) \in \mathrm{cl}_\mu\, \Xi_\Delta.$$

Proof. To begin with, let us show that the constrained minimization problem $\left\langle\inf_{(u,y)\in\mathrm{cl}_\mu\, \Xi_\Delta} J_0(u, y)\right\rangle$ has a nonempty set of solutions. Indeed, in view of definition of the set $\mathrm{cl}_\mu\, \Xi_\Delta$ (see Lemma 5.1), there exists a sequence $\left\{(u_k, y_k)\right\}_{k\in\mathbb{N}} \subset \mathrm{cl}_\mu\, \Xi_\Delta$ such that

$$\lim_{k\to\infty} J_0(u_k, y_k) = \inf_{(u,y)\in\mathrm{cl}_\mu\, \Xi_\Delta} J_0(u, y). \quad\quad\quad (5.33)$$

Moreover, because of the density of $C^\infty(\overline{\Omega}) \times C_0^\infty(\Omega)$ in $L^p(\Omega) \times H_0^1(\Omega)$, we can suppose that the sequence is rather regular. For our purpose it is enough to have

the following inclusion: $(u_k, y_k) \in \Xi_\Delta$ for all $k \in \mathbb{N}$. Then, there exists a constant $C > 0$ such that

$$\sup_{k \in \mathbb{N}} \|u_k\|^p_{L^p(\Omega)} \le 2^{p-1} \|g\|^p_{L^p(\Omega)} + 2^{p-1} \sup_{k \in \mathbb{N}} \|g - u_k\|^p_{L^p(\Omega)}$$

$$\le 2^{p-1} \|g\|^p_{L^p(\Omega)} + 2^{p-1} p \sup_{k \in \mathbb{N}} J_0(u_k, y_k)$$

$$\overset{\text{by (5.33)}}{\le} 2^{p-1} \|g\|^p_{L^p(\Omega)} + 2^{p-1} p C^*,$$

$$\sup_{k \in \mathbb{N}} \|y_k\|_{H_0^1(\Omega)} \overset{\text{by (3.7)}}{\le} C_4 \sup_{k \in \mathbb{N}} \|u_k\|_{L^p(\Omega)} + C_5 < +\infty.$$

So, the minimizing sequence $\{(u_k, y_k)\}_{k \in \mathbb{N}}$ is relatively μ-compact in $L^p(\Omega) \times H_0^1(\Omega)$, that is, there exists a pair $(u^*, y^*) \in L^p(\Omega) \times H_0^1(\Omega)$ such that, up to a subsequence,

$$(u_k, y_k) \overset{\mu}{\longrightarrow} (u^*, y^*) \quad \text{and} \quad (u^*, y^*) \overset{\text{by Proposition 3.2 and Lemma 5.1}}{\in} \mathrm{cl}_\mu \, \Xi_\Delta.$$

To conclude the optimality of (u^*, y^*) to the problem $\big\langle \inf_{(u,y) \in \mathrm{cl}_\mu \Xi_\Delta} J_0(u, y) \big\rangle$, it remains to make use of the lower semi-continuity of the cost functional $J_0 : \mathrm{cl}_\mu \, \Xi_\Delta \to \mathbb{R}$ with respect to the μ-convergence.

We are now in a position to prove the equality (5.32). By contraposition, let us assume that there exists a pair $(u^*, y^*) \in \mathrm{cl}_\mu \, \Xi_\Delta$ such that

$$J_0(u^0, y^0) > \inf_{(u,y) \in \mathrm{cl}_\mu \Xi_\Delta} J_0(u, y) = J_0(u^*, y^*). \tag{5.34}$$

Then, due to Proposition 5.1, we can construct a sequence $\{(\widehat{u}_\varepsilon, \widehat{y}_\varepsilon)\}_{\varepsilon > 0}$ in $L^p(\Omega) \times H_0^1(\Omega)$ with properties

$$(\widehat{u}_\varepsilon, \widehat{y}_\varepsilon) \overset{\mu}{\longrightarrow} (u^*, \widehat{y}) \ \text{ in } \ L^p(\Omega) \times H_0^1(\Omega), \quad (\widehat{u}_\varepsilon, \widehat{y}_\varepsilon) \in \Xi_\Delta, \quad \forall \, \varepsilon > 0, \tag{5.35}$$

$$\widehat{u}_\varepsilon \to u^* \text{ strongly in } L^p(\Omega), \tag{5.36}$$

$$\Delta \widehat{y}_\varepsilon \in L^2(\Omega) \quad \forall \, \varepsilon > 0, \quad \text{and} \quad \lim_{\varepsilon \to 0} \left(\frac{\varepsilon}{2} \|\Delta \widehat{y}_\varepsilon\|^2_{L^2(\Omega)} \right) = C_\Delta, \tag{5.37}$$

where the constant C_Δ is defined by (5.30). Moreover, since a weak solution to the boundary value problem

$$-\Delta y = F'(y) + u^* \ \text{ in } \ \Omega, \qquad y = 0 \ \text{ on } \ \partial\Omega,$$

can be non-unique, we admit the case that $y^* \ne \widehat{y}$ as elements of $H_0^1(\Omega)$. Then we can write down

$$J_\varepsilon(u^0_\varepsilon, y^0_\varepsilon) = \inf_{(u,y) \in \Xi_\Delta} J_\varepsilon(u, y) \le J_\varepsilon(\widehat{u}_\varepsilon, \widehat{y}_\varepsilon), \quad \forall \, \varepsilon > 0,$$

or in other terms

$$\frac{\varepsilon}{2} \int_\Omega |\Delta y^0_\varepsilon|^2 \, dx + \frac{1}{p} \int_\Omega |g - u^0_\varepsilon|^p \, dx \le \frac{\varepsilon}{2} \int_\Omega |\Delta \widehat{y}_\varepsilon|^2 \, dx + \frac{1}{p} \int_\Omega |g - \widehat{u}_\varepsilon|^p \, dx. \tag{5.38}$$

Passing to the limit in both sides of this inequality as $\varepsilon \to 0$ and taking into account that

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{2} \int_{\Omega} |\Delta y_{\varepsilon}^{0}|^2 \, dx \overset{\text{by } (5.31)_1}{=} C_{\Delta},$$

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{2} \int_{\Omega} |\Delta \widehat{y}_{\varepsilon}|^2 \, dx \overset{\text{by } (5.37)}{=} C_{\Delta},$$

$$\liminf_{\varepsilon \to 0} \int_{\Omega} |g - u_{\varepsilon}^{0}|^p \, dx \overset{\text{by } (5.29)}{\geq} \int_{\Omega} |g - u^0|^p \, dx,$$

$$\lim_{\varepsilon \to 0} \int_{\Omega} |g - \widehat{u}_{\varepsilon}|^p \, dx \overset{\text{by } (5.36)}{=} \int_{\Omega} |g - u^*|^p \, dx,$$

we arrive at the relation

$$J_0(u^0, y^0) = \frac{1}{p} \int_{\Omega} |g - u^0|^p \, dx \leq \frac{1}{p} \int_{\Omega} |g - u^*|^p \, dx = J_0(u^*, \widehat{y}) \equiv J_0(u^*, y^*),$$

which comes into conflict with (5.34). The proof is complete. $\square$

Remark 5.2. Because of ill-posedness of the original BVP (1.1)–(1.2), it is reasonably to suppose that the set of solutions to minimization problem (5.32) is not singleton. On the other hand, we have

$$\inf_{(u,y) \in \text{cl}_\mu \Xi_\Delta} J_0(u, y) = \frac{1}{p} \inf_{(u,y) \in \text{cl}_\mu \Xi_\Delta} \left[\|g - u\|_{L^p(\Omega)}^p \right] = \frac{1}{p} \inf_{u \in \overline{\Lambda}} \left[\|g - u\|_{L^p(\Omega)}^p \right].$$

It means that the minimal value of the cost functional $J_0(u^0, y^0)$ does not depend on the y-component of the optimal pair. Hence, Theorem 5.1 implies that even if the sequence of optimal solutions $\left\{ (u_{\varepsilon}^{0}, y_{\varepsilon}^{0}) \right\}_{\varepsilon > 0}$ to the fictitious problems (4.1) has more than one μ-cluster pair, their u-components must coincide. In other words, if $\left\{ (u_{\varepsilon}^{0}, y_{\varepsilon}^{0}) \right\}_{\varepsilon > 0}$ is a sequence of optimal pairs to the fictitious problems (4.1), then $\left\{ y_{\varepsilon}^{0} \right\}_{\varepsilon > 0}$ is relatively weakly compact in $H_0^1(\Omega)$ and there exists a unique $u^0 \in \overline{\Lambda}$ such that

$$u_{\varepsilon}^{0} \rightharpoonup u^0 \quad \text{in} \quad L^p(\Omega) \quad \text{as} \quad \varepsilon \to 0,$$

where u^0 is a minimizer to the minimization problem $\frac{1}{p} \inf_{u \in \overline{\Lambda}} \left[\|g - u\|_{L^p(\Omega)}^p \right]$. This circumstance and ill-posedness of the BVP (1.1)–(1.2) motivates us to introduce the following concept.

Definition 5.1. Let g be a given element of $L^p(\Omega)$ with $p > 2$. Then we say that a distribution $y^* = y^*(g) \in H_0^1(\Omega)$ is the approximate solution to the boundary value problem (1.1)–(1.2) if $y^* \in Y$ and y^* satisfies the relations

$$\int_{\Omega} (\nabla y^*, \nabla \varphi) \, dx = \int_{\Omega} F'(y^*) \varphi \, dx + \int_{\Omega} g^* \varphi \, dx, \quad \forall \varphi \in C_0^\infty(\Omega), \qquad (5.39)$$

$$\int_{\Omega} |\nabla y^*|^2 \, dx \leq \frac{2N}{N-2} \int_{\Omega} (F(y^*) - F(0)) \, dx$$

$$- \frac{2}{N-2} \int_{\Omega} g^* \, (x - x_0, \nabla y^*) \, dx, \qquad (5.40)$$

where

$$\|g - g^*\|_{L^p(\Omega)}^p = \inf_{(u,y) \in \mathrm{cl}_\mu \, \Xi_\Delta} \|g - u\|_{L^p(\Omega)}^p. \tag{5.41}$$

As immediately follows from this definition, an approximate solution y^* is not unique, in general. Moreover, y^* coincides with a weak solution to (1.1)–(1.2) in the sense of Definition 2.1 provided $g^* = g$. However, in this case we have a weak solution with an extra property: this solution satisfies the inequality (3.1) even if we do not know whether $\Delta y^* \in L^2(\Omega)$. In the context of Definition 5.1, it arises the question about existence and attainability of the approximate solutions to the boundary value problem (1.1)–(1.2). In view of this, it makes sense to give the following final result which is an obvious consequence of Theorem 5.1, Proposition 5.2, and Theorem 4.2.

Theorem 5.2. *Let Ω be a bounded open subset of $\mathbb{R}^N$ ($N > 2$) which is assumed to be star-shaped with respect to some of its interior point x_0. Let $F : \mathbb{R} \to (0, +\infty)$ be a mapping of the class $C^1_{loc}(\mathbb{R})$ such that F satisfies estimate (2.1) and its derivative $F' : \mathbb{R} \to (0, \infty)$ is a strictly convex function. Let $g \in L^2(\Omega)$ be a given distribution. Assume that there exists a positive value δ such that $g \in L^{2+\delta}(\Omega)$. Then the set of approximate solutions to the boundary value problem (1.1)–(1.2) is non-empty for the given g. Moreover, in this case some of such solutions $y^* \in Y$ can be attained as follows: y^* is an $H^1_0(\Omega)$-weak cluster point of the sequence $\{y_\varepsilon^0\}_{\varepsilon>0}$, where $\{(u_\varepsilon^0, y_\varepsilon^0)\}_{\varepsilon>0}$ are minimizers to the corresponding fictitious OCPs (4.1).*

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