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MATHEMATICAL MODELING OF PARKINSON'S ILLNESS BY CHAOTIC DYNAMICS METHODS

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An modeling attempt of behavior process of brain electric impulses for some patient by the solutions of 3D system of quadratic differential equations is undertaken. (This system of differential equations was got from a multivariate times series with the help of polynomial averages and least square method.) Transition conditions from a chaotic attractor to a limit cycle (and vice versa) of the system of differential equations are found. Exactly these conditions characterize beginning of process of disease by Parkinson's illness at the patient.

Keywords: time series, 3D system of quadratic differential equations, algebraic invariant, limit cycle, chaotic attractor, brain electric activity, electroencephalogram.

1. Introduction

Last years theory of chaos, a nonlinear dynamics, and sciences about complication of one or another processes began to act important role in biology, medicine and row of contiguous fields. Application of chaos in medicine does not allow to do prognoses and decide some private tasks. Nevertheless, the theory of chaos allows rather to describe some aspects of behavior of the complex biological systems by certain numerical descriptions, such as the Lyapunov exponent, fractal dimension, multiplicity of limit cycle etc. By other words, the theory of chaos can be used for classification of the states of organism. Thus, most valuable achievement will be not got some numerical values, but description and reformulation medical problems in terms of simulation tasks and measurement of signals [8].

One of important examples of such approach are epileptology methods. These methods being based on the study of brain electrical activity with the help of electroencephalograms (EEG). From the experimental point of view a problem consists in that on the basis of time series of the measured values of rhythms of brain activity to recreate development of the dynamic system (it is a brain) in phase space. Further with the help of the got dynamic system it is necessary to study processes resulting in appearance epilepsy or Parkinson's illness [6, 10].

Parkinson's illness is characterized by a recurrent and sudden malfunction of the brain that is termed seizure. Sickly seizures reflect the clinical signs of an excessive and hypersynchronous activity of neurons in the cerebral cortex. Depending on the extent of involvement of other brain areas during the course of the seizure, types Parkinson's illness can be divided into two main classes. Generalized seizures involve almost the entire brain while focal (or partial) seizures originate from a circumscribed region of the brain and remain restricted to this region [10].

In the present paper we consider the problem of reconstructing dynamical system (it is a system of differential equations describing impulses of brain activity) from multivariate time series.

2. Mathematical statement of problem and its discussion

We will assume that we can measure the rhythms $z_1(t_i), ..., z_n(t_i), i = 1, 2, ..., N$, of cerebral activity in n points of cerebral cortex with the help of EEG. We also suppose that these measurements are noisy. Thus, we have multivariate time series

$$z_1(t_i) = x_1(t_i) + \theta_1(t_i), ..., z_n(t_i) = x_n(t_i) + \theta_n(t_i), \tag{2.1}$$

which defined for $\forall t_i \in (t_1, t_N)$. Here $\forall i = 1, 2, ..., N$, we have $t_i = i\Delta t$ and $\Delta t = (t_N - t_1)/N$. In addition, we suppose that $\theta_1(t_i), ..., \theta_n(t_i)$ are Gaussian (white) noises, unable by definition to produce statistically systematical errors [7, 11].

Finally, we assume that $x_1(t_i), ..., x_n(t_i)$ is a discrete approximation of some n-dimensional curve $\mathbf{x}(t) = (x_1(t), ..., x_n(t))^T \in \mathbb{R}^n$ [5].

Principal problem. *Construct the quadratic system of differential equations*

$$\begin{cases} \dot{x}_1(t) = \sum\limits_{j=1}^{n} a_{1j} x_j(t) + \mathbf{x}^T(t) B_1 \mathbf{x}(t) + c_1 \equiv f_1(\mathbf{x}(t)), \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad , \\ \dot{x}_n(t) = \sum\limits_{j=1}^{n} a_{nj} x_j(t) + \mathbf{x}^T(t) B_n \mathbf{x}(t) + c_n \equiv f_n(\mathbf{x}(t)) \end{cases} \tag{2.2}$$

such that there exists bounded solution $\mathbf{x}(t)$ *(*$\lim_{t\to\infty} \|\mathbf{x}(t)\| < \infty$*) of this system, which approximates the time-variate series (2.1) with given accuracy in the set points* $t_1, ..., t_N$ *at any choice of the vector of initial values* $\mathbf{x}(0) = (x_{10}, ..., x_{n0})^T$.

In connection with the indicated principal problem there is the following question: whether or not there exists an n-th order $(n > 1)$ dynamical system (2.2) having as solution the bounded function $\mathbf{x}(t)$?

Introduce a few definitions.

Definition 2.1. The trajectory $\mathbf{x}(t)$ is called simple if there are no points transversal or tangential self-intersection on it.

Definition 2.2. The trajectory $\mathbf{x}(t)$ is called regular if $\forall t \in (t_1, t_N)$ $\dot{\mathbf{x}}(t) \neq \mathbf{0}$.

Definition 2.3. The trajectory $\mathbf{x}(t)$ is called elementary if it is one-to-one function and one-to-one continuous. The trajectory $\mathbf{x}(t)$ is called an embedding if it is both elementary and regular.

Theorem 2.1. *[7, 11] Let $\mathbf{x}(t)$ be a real-valued and analytic function defined on the interval (t_1, t_N) such that the curve $\mathbf{x}(t)$ is simple and regular. Then there exists a real-valued analytic function $\mathbf{F}(\mathbf{x}) \in \mathbb{R}^n$ such that $\mathbf{x}(t)$ is a solution of system $\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x})$.*

Theorem 2.2. *[7, 11] Let $\mathbf{F}(\mathbf{x}) \in \mathbb{R}^n$ be the function defined in Theorem (2.1). Then every solution of the system $\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x})$ is either an equilibrium point or closed trajectory (it may be a limit cycle or torus) or an embedding (elementary and regular) one.*

The practical procedure for determining the elementary and regularity of the function $\mathbf{x}(t)$ is given in [7,11]. However, it should be noted that if $t_N \to \infty$, then theorems (2.1),(2.2) are not applicable. Thus, an application of these theorems for the prediction tasks becomes problematical.

Further, we use the procedure for determining unknown quadratic right sides of the system of differential equations (2.2), which was suggested in [7, 11]. This procedure is based on the least square method and the fact that we know sufficient precision the components of $\mathbf{x}(t)$ and its derivative $\dot{\mathbf{x}}(t)$.

We will use the following designations: $\mathbf{x}(t_i) = (x_1(t_i), x_2(t_i), ..., x_n(t_i))^T = (x_{1i}, x_{2i}, ..., x_{ni})^T$, $\dot{\mathbf{x}}(t_i) = (\dot{x}_1(t_i), \dot{x}_2(t_i), ..., \dot{x}_n(t_i))^T = (\dot{x}_{1i}, \dot{x}_{2i}, ..., \dot{x}_{ni})^T$, where $\dot{x}_{ki} = (x_{k,i+1} - x_{ki})/\Delta t$; $k = 1, ..., n$; $i = 0, 1, ..., N$.

Introduce the matrix of unknown coefficients of system (2.2):

$$Y = \begin{pmatrix} c_1 & a_{11} & \cdots & a_{1n} & b^1_{11} & \cdots & b^1_{nn} & 2b^1_{12} & \cdots & 2b^1_{n-1,n} \\ \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\ c_n & a_{n1} & \cdots & a_{nn} & b^n_{11} & \cdots & b^n_{nn} & 2b^n_{12} & \cdots & 2b^n_{n-1,n} \end{pmatrix} \in \mathbb{R}^{n\times m},$$

where $m = 1 + 2n + n(n-1)/2$.

Introduce also $(N \times m)$-matrix

$$X = \begin{pmatrix} 1 & x_{11} & \cdots & x_{n1} & x^2_{11} & \cdots & x^2_{n1} & x_{11}x_{21} & \cdots & x_{n-1,1}x_{n,1} \\ \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\ 1 & x_{1N} & \cdots & x_{nN} & x^2_{1N} & \cdots & x^2_{nN} & x_{1N}x_{2N} & \cdots & x_{n-1,N}x_{n,N} \end{pmatrix}$$

and $(N \times n)$-matrix

$$\dot{X} = \begin{pmatrix} \dot{x}_{11} & \cdots & \dot{x}_{n1} \\ \vdots & \ldots & \vdots \\ \dot{x}_{1N} & \cdots & \dot{x}_{nN} \end{pmatrix},$$

elements of which are known. Then by the least square method [7, 11], we have $Y^T = (X^T X)^{-1} X^T \dot{X}^T$. Further, the following is said in work [7]: In view of the fact that number N may be chosen arbitrary large, a high precision reconstruction may be achieved. Thus, we can expected that the solution of reconstructed system will be near the purified solution $\mathbf{x}(t)$.

However, it should be said that one important circumstance, which can arise up at a reconstruction, remained outside attention of authors of article [7]. The point is that in [7] it is assumed that the interval (t_1, t_N) is finite. If the problem of long-term prediction is considered, it is necessary to assume that $t_N \to \infty$. In this case a reconstruction must be fulfilled so that system (2.2) had the bounded solutions. Exactly to the question of existence of the bounded solutions in the system (2.2) the next section will be devoted.

3. Existence conditions of bounded solutions in 3D systems of quadratic differential equations

Introduce the Cartesian product of the real linear spaces by the following formula:

$$\mathbb{R}^K = \mathbb{R}^n \times \mathbb{R}^{n\times n} \times \underbrace{\mathbb{R}^{n(n+1)/2} \times ... \times \mathbb{R}^{n(n+1)/2}}_{n},\ K = n + n^2 + (n+1)\,n^2/2.$$

Thus, $\mathbb{R}^K$ is a real linear space, elements of which are K-dimensional vectors of coefficients $(\mathbf{c}, A, B_1, ..., B_n)$ of system (2.2).

Let $\mathbb{V} \subset \mathbb{R}^K$ be an arbitrary nonempty open set. (Thus, we have $\overline{\mathbb{V}} = \mathbb{R}^K$, where $\overline{\mathbb{V}}$ is the closure of $\mathbb{V}$.)

Definition 3.1. . System (2.2) is called a generic system if the vector $(\mathbf{c}, A, B_1, ..., B_n) \in \mathbb{V}$.

Let $\mathbb{W} \subset \mathbb{C}^n$ be an algebraic variety of all complex solutions of the following system of algebraic equations: $f_1(\mathbf{x}) = 0, ..., f_n(\mathbf{x}) = 0$ [9]. (By virtue of the definition of the functions $f_1(\mathbf{x}), ..., f_n(\mathbf{x})$ the variety $\mathbb{W}$ is a finite nonzero point set [9].)

Definition 3.2. . If the variety $\mathbb{W}$ contains even one real point, then system (2.2) is called a system with equilibriums; otherwise system (2.2) is called a system without equilibriums.

Define by $\mathbb{C}l_{\mathbb{R}}$ ($\mathbb{C}l_{\mathbb{C}}$) all systems of type (2.2) with equilibriums (without equilibriums). It is clear that $\mathbb{R}^K = \mathbb{C}l_{\mathbb{R}} \cup \mathbb{C}l_{\mathbb{C}}$ and $\mathbb{C}l_{\mathbb{R}} \cap \mathbb{C}l_{\mathbb{C}} = \varnothing$.

3.1. Invariants of 2D Autonomous Quadratic Systems

In this section, we suppose that the generic system (2.2) is a system of the class $\mathbb{C}l_{\mathbb{R}}$. In addition, we put that $n = 2$. (By suitable replacements of variables it is always possible to obtain that in system (2.2) $\mathbf{c} = (c_1, ...c_n)^T = 0$. Therefore, we will consider that $\mathbf{c} = 0$.)

Consider the following 2D autonomous quadratic system:

$$\begin{cases} \dot{x}(t) = a_{11}x(t) + a_{12}y(t) + b_{11}x^2(t) + 2b_{12}x(t)y(t) + b_{22}y^2(t), \\ \dot{y}(t) = a_{21}x(t) + a_{22}y(t) + c_{11}x^2(t) + 2c_{12}x(t)y(t) + c_{22}y^2(t), \end{cases} \tag{3.1}$$

where $a_{11}, ..., a_{22}, b_{11}, ..., b_{22}, c_{11}, ..., c_{22}$ are real numbers. (A multiplier $2b_{12}$ (or $2c_{12}$) before $x(t)y(t)$ is represented in such form with the purpose of simplification of calculations of invariants. If there is no necessity in such calculations, we will write b_{12} (or c_{12}).)

Introduce the following real (2×2)-matrices:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \ T_1 = \begin{pmatrix} b_{11} & b_{12} \\ c_{11} & c_{12} \end{pmatrix}, \ T_2 = \begin{pmatrix} b_{12} & b_{22} \\ c_{12} & c_{22} \end{pmatrix}. \tag{3.2}$$

Replace variables x, y in system (3.1) by new variables x_1, y_1 under the formula

$$\begin{pmatrix} x \\ y \end{pmatrix} \to S \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix},$$

where S is a linear transformation from the group $GL(2,\mathbb{R})$ of all linear inverse transformations of the space $\mathbb{R}^2$ [1]. In this case the triple of matrices (A, T_1, T_2) transforms into triple

$$S \circ (A, T_1, T_2) = (S^{-1}AS, (S^{-1}T_1, S^{-1}T_2) \cdot (S \otimes S)) = (A_s, T_{1s}, T_{2s}).$$

Remember that a scalar polynomial $f(A, T_1, T_2)$ is called an invariant of weight l of the group $GL(2,\mathbb{R})$, if $\forall S \in GL(2,\mathbb{R})$ and $\forall (A, T_1, T_2)$ $\ f(S \circ (A, T_1, T_2)) = f(A_s, T_{1s}, T_{2s}) = (\det S)^l \times f(A, T_1, T_2))$, where $l \geq 0$ is some integer [1].

With the help of matrices T_1, T_2, we construct the auxiliary not depending on A invariants of weight 2 [1]:

$$I_1 = \det \begin{pmatrix} (\operatorname{tr} T_1, \operatorname{tr} T_2) \cdot T_1 \\ (\operatorname{tr} T_1, \operatorname{tr} T_2) \cdot T_2 \end{pmatrix}, \quad J_2 = \det(T_1T_2 - T_2T_1),$$

$$K_3 = \det \begin{pmatrix} \operatorname{tr} T_1, \operatorname{tr} T_2 \\ (\operatorname{tr} T_1, \operatorname{tr} T_2) \cdot (T_1T_2 - T_2T_1) \end{pmatrix},$$

where $\operatorname{tr} P$ is a trace of the square matrix P.

Now we can introduce the main invariants of the present paper:

$$L = I_1 - J_2 - K_3, \ D = I_1 + 27J_2 - 5K_3 \tag{3.3}$$

of weight 2 [1]. (It is easy to check that $\deg L = \deg D = 4$.)

3.2. Case $L < 0$, $D < 0$.

By suitable linear replacements of the variables x and y system (3.1) can be resulted to the following aspect:

$$\begin{cases} \dot{x}(t) = a_{11}x(t) + a_{12}y(t) + b_{11}x^2(t) + b_{22}y^2(t), \\ \dot{y}(t) = a_{21}x(t) + a_{22}y(t) + 2c_{12}x(t)y(t) + c_{22}y^2(t). \end{cases} \tag{3.4}$$

(For simplicity we have left the former designations of variables x and y, and corresponding coefficients.)

Compute the invariants L and D for system (3.4). Then we have:

$$L = b_{11}(b_{11}c_{22}^2 + 4b_{22}c_{12}^2),\ D = -(b_{11} - 2c_{12})^2(4b_{11}b_{22} - c_{22}^2 - 8b_{22}c_{12}). \quad (3.5)$$

Let the conditions $L < 0$ and $D < 0$ be satisfied. Then from (3.5) it follows that $b_{11}b_{22} < 0$ and $b_{22}c_{12} < 0$.

Without loss of generality, we will consider that $b_{11} < 0$, $b_{22} > 0$, $c_{12} < 0$. In this case, we can do the replacements of variables $x \to x/(-2c_{12})$ and $y \to y/\sqrt{-2c_{12}b_{22}}$. Then system (3.4) can be represented in the form:

$$\begin{cases} \dot{x}(t) = a_{11}x(t) + a_{12}y(t) + b_{11}x^2(t) + y^2(t), -0.5 < b_{11} < 0, \\ \dot{y}(t) = a_{21}x(t) + a_{22}y(t) - x(t)y(t) + c_{22}y^2(t),\ |c_{22}| < \sqrt{2}. \end{cases} \quad (3.6)$$

(Here the conditions $L < 0,\ D < 0$ were used. For simplicity, we again have left the former designations of variables x and y, and corresponding coefficients.)

Further, we will use the following theorem.

Theorem 3.1. *[1] Let $L < 0$, $D < 0$, and the point $(0,0)$ be stable or saddle. Suppose also that there doesn't exist a real eigenvector $(v_1, v_2)^T$ of the matrix A such that*

$$\begin{cases} b_{11}v_1^2 + 2b_{12}v_1v_2 + b_{22}v_2^2 = kv_1, \\ c_{11}v_1^2 + 2c_{12}v_1v_2 + c_{22}v_2^2 = kv_2;\ k \in \mathbb{R}. \end{cases}$$

Then there exists an open domain $\mathbb{W} \subset \mathbb{R}^2$ such that $\forall (x_0, y_0) \in \mathbb{W}$ the solutions $x(t) = x(x_0, y_0, t)$, $y(t) = y(x_0, y_0, t)$ of system (3.1) (or (3.4)) are bounded.

Note that if $L \cdot D \leq 0$, then there doesn't exist initial values x_0, y_0 such that the solutions $x(t) = x(x_0, y_0, t)$, $y(t) = y(x_0, y_0, t)$ of system (3.1) (or (3.4)) are bounded. (The case $L = 0$ considered in [3]. However at the modeling of the real processes situation $L \leq 0$ more widespread. Therefore, we suppose that $L \leq 0$.)

3.3. Existence Conditions of Chaos in System (2.2)

In this section we will consider that the system of algebraic equations $f_1(\mathbf{x}) = \ldots = f_n(\mathbf{x}) = 0$ has a real solution $\xi = (\xi_1, ..., \xi_n)^T$. (The point ξ is an equilibrium point of system (2.2).)

Introduce a new vector variable $\mathbf{y} = (y_1, ..., y_n)^T$, which is given by the formula $\mathbf{x} = \mathbf{y} + \xi$. Then system (2.2) can be represented in the form

$$\begin{cases} \dot{y}_1(t) = [(a_{11}, ..., a_{1n}) + 2\xi^T B_1]\mathbf{y}(t) + \mathbf{y}^T(t)B_1\mathbf{y}(t), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ \dot{y}_n(t) = [(a_{n1}, ..., a_{nn}) + 2\xi^T B_n]\mathbf{y}(t) + \mathbf{y}^T(t)B_n\mathbf{y}(t). \end{cases} \quad (3.7)$$

Having fulfilled the change of variables we can return to previous variable $\mathbf{x}$.

Let $n = 3$. Then instead of system (3.7), we will consider the following 3D real autonomous system

$$\dot{\mathbf{x}}(t) = H\mathbf{x} + \mathbf{f}(\mathbf{x}), \quad (3.8)$$

where $\mathbf{x} = (x, y, z)^T$; $H = \{h_{ij}\}$, $i, j = 1, ..., 3$, is a real (3×3)-matrix;

$$\mathbf{f}(\mathbf{x}) = (f_1(x, y, z), f_2(x, y, z), f_3(x, y, z))^T \in \mathbb{R}^3$$

and

$$\begin{aligned} f_1(x, y, z) &= a_{11}x^2 + a_{12}xy + a_{22}y^2 + a_{13}xz + a_{23}yz + a_{33}z^2, \\ f_2(x, y, z) &= b_{11}x^2 + b_{12}xy + b_{22}y^2 + b_{13}xz + b_{23}yz + b_{33}z^2, \\ f_3(x, y, z) &= c_{11}x^2 + c_{12}xy + c_{22}y^2 + c_{13}xz + c_{23}yz + c_{33}z^2 \end{aligned}$$

are real quadratic polynomials.

Introduce into system (3.8) new variables ρ and ϕ under the formulas: $y = \rho\cos\phi$, $z = \rho\sin\phi$, where $\rho > 0$. Then, after replacement of variables and multiplication of the second and third equations of system (3.8) on the matrix

$$\begin{pmatrix} \cos\phi(t) & \sin\phi(t) \\ -(\sin\phi(t))/\rho(t) & (\cos\phi(t))/\rho(t) \end{pmatrix},$$

we get

$$\left\{ \begin{aligned} \dot{x}(t) = {} & h_{11}x(t) + [h_{12}\cos\phi(t) + h_{13}\sin\phi(t)]\rho(t) + a_{11}x^2(t) \\ & +[a_{12}\cos\phi(t) + a_{13}\sin\phi(t)]x(t)\rho(t) \\ & +[a_{22}\cos^2\phi(t) + a_{23}\cos\phi(t)\sin\phi(t) + a_{33}\sin^2\phi(t)]\rho^2(t), \\ \dot{\rho}(t) = {} & [h_{21}\cos\phi(t) + h_{31}\sin\phi(t)]x(t) + [h_{22}\cos^2\phi(t) + h_{33}\sin^2\phi(t) \\ & +(h_{32} + h_{23})\cos\phi(t)\sin\phi(t)]\rho(t) + [b_{11}\cos\phi(t) + c_{11}\sin\phi(t)]x^2(t) \\ & +[b_{12}\cos^2\phi(t) + (b_{13} + c_{12})\cos\phi(t)\sin\phi(t) + c_{13}\sin^2\phi(t)]x(t)\rho(t) \\ & +[b_{22}\cos^3\phi(t) + (b_{23} + c_{22})\cos^2\phi(t)\sin\phi(t) \\ & +(b_{33} + c_{23})\cos\phi(t)\sin^2\phi(t) + c_{33}\sin^3\phi(t)]\rho^2(t), \\ \dot{\phi}(t) = {} & [-h_{21}\sin\phi(t) + h_{31}\cos\phi(t)]\frac{x(t)}{\rho(t)} + [h_{32}\cos^2\phi(t) - h_{23}\sin^2\phi(t) \\ & +(h_{33} - h_{22})\cos\phi(t)\sin\phi(t)] + [c_{11}\cos\phi(t) - b_{11}\sin\phi(t)]\frac{x^2(t)}{\rho(t)} \\ & -[b_{13}\sin^2\phi(t) + (b_{12} - c_{13})\sin\phi(t)\cos\phi(t) - c_{12}\cos^2\phi(t)]x(t) \\ & -[-c_{22}\cos^3\phi(t) + (b_{22} - c_{23})\cos^2\phi(t)\sin\phi(t) \\ & +(b_{23} - c_{33})\cos\phi(t)\sin^2\phi(t) + b_{33}\sin^3\phi(t)]\rho(t). \end{aligned} \right. \tag{3.9}$$

Consider the system

$$\left\{ \begin{aligned} \dot{x}(t) = {} & s_{11}(\cos\phi, \sin\phi)x + s_{12}(\cos\phi, \sin\phi)\rho \\ & +p_{11}(\cos\phi, \sin\phi)x^2 + p_{12}(\cos\phi, \sin\phi)x\rho + p_{22}(\cos\phi, \sin\phi)\rho^2, \\ \dot{\rho}(t) = {} & s_{21}(\cos\phi, \sin\phi)x + s_{22}(\cos\phi, \sin\phi)\rho \\ & +q_{11}(\cos\phi, \sin\phi)x^2 + q_{12}(\cos\phi, \sin\phi)x\rho + q_{22}(\cos\phi, \sin\phi)\rho^2, \end{aligned} \right. \tag{3.10}$$

where ϕ is a real parameter and

$s_{11}(\cos\phi, \sin\phi) = h_{11}$, $s_{12}(\cos\phi, \sin\phi) = h_{12}\cos\phi + h_{13}\sin\phi$,
$s_{21}(\cos\phi, \sin\phi) = h_{21}\cos\phi + h_{31}\sin\phi$,
$s_{22}(\cos\phi, \sin\phi) = h_{22}\cos^2\phi + h_{33}\sin^2\phi + (h_{32} + h_{23})\cos\phi\sin\phi$,

$$\begin{aligned}
&p_{11}(\cos\phi,\sin\phi) = a_{11}, p_{12}(\cos\phi,\sin\phi) = a_{12}\cos\phi + a_{13}\sin\phi,\\
&p_{22}(\cos\phi,\sin\phi) = a_{22}\cos^2\phi + a_{23}\cos\phi\sin\phi + a_{33}\sin^2\phi,\\
&q_{11}(\cos\phi,\sin\phi) = b_{11}\cos\phi + c_{11}\sin\phi,\\
&q_{12}(\cos\phi,\sin\phi) = b_{12}\cos^2\phi + (b_{13}+c_{12})\cos\phi\sin\phi + c_{13}\sin^2\phi,\\
&q_{22}(\cos\phi,\sin\phi) = b_{22}\cos^3\phi + (b_{23}+c_{22})\cos^2\phi\sin\phi + (b_{33}+c_{23})\cos\phi\sin^2\phi +\\
&\qquad c_{33}\sin^3\phi.
\end{aligned}$$

Notice that the replacements of cartesian coordinates by polar is needed so that in system (3.10) both equations would be nonlinear with respect to the unknowns x and ρ. (The equation $\dot{\phi}(t) = ...$ in system (3.10) is not included.)

Let $v(\phi) = (v_1(\phi), v_2(\phi))^T$ be an arbitrary eigenvector of the linear operator

$$S(\phi) = \begin{pmatrix} s_{11}(\cos\phi,\sin\phi) & s_{12}(\cos\phi,\sin\phi) \\ s_{21}(\cos\phi,\sin\phi) & s_{22}(\cos\phi,\sin\phi) \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2,$$

depending on the parameter $\phi \in \mathbb{R}$.

Let $p_{ij}(\phi) \equiv p_{ij}(\cos\phi,\sin\phi),\ q_{ij}(\phi) \equiv q_{ij}(\cos\phi,\sin\phi), i,j = 1,2$. Define by

$$G(v(\phi)) = \det\begin{pmatrix} v_1(\phi) & p_{11}(\phi)v_1^2(\phi) + p_{12}(\phi)v_1(\phi)v_2(\phi) + p_{22}(\phi)v_2^2(\phi) \\ v_2(\phi) & q_{11}(\phi)v_1^2(\phi) + q_{12}(\phi)v_1(\phi)v_2(\phi) + q_{22}(\phi)v_2^2(\phi) \end{pmatrix}$$

the bounded real function.

For system (3.10)) we write the matrices $A(\phi)$, $T_1(\phi)$ and $T_2(\phi)$ are given by formulas (3.2). We also compute the invariants $L(\phi)$ and $D(\phi)$ are given by formulas (3.3).

Theorem 3.2. *Suppose that $\forall\phi \in \mathbb{R}$ the following conditions for system (3.10) are fulfilled:*

(i) $G(v(\phi)) \not\equiv 0$;

(ii) $L(\phi) < 0$, $D(\phi) < 0$

(iii) either $\det S(\phi) \le 0$ or $\det S(\phi)$ is a periodic alternating in sign on the interval $(-\infty,\infty)$ function.

Then there exists the open domain $\mathbb{V} \subset \mathbb{R}^3$ such that $\forall(x_0, y_0, z_0) \in \mathbb{V}$ the solutions $x(t), y(t), z(t)$ of system (3.8) are bounded. In addition, there can exist a limit cycle in system (3.8).

Proof. It is shown in [1] that if $S(\phi) \equiv 0$, then at any signs of functions $L(\phi)$ and $D(\phi)$ always there are unbounded solutions of system (3.10). The solutions of system (3.10) will be also unbounded, if an fixed point of the operator of quadratic part of system (3.10) is the eigenvector of the matrix $S(\phi)$ [1, 3]. In order to eliminate a similar situation the condition (i) was introduced.

Define by

$$\mathbb{W} := \{\phi \in \mathbb{R} |\ L(\phi) < 0,\ D(\phi) < 0\}$$

a nonempty open set in $\mathbb{R}$.

It is known that for arbitrary point $\phi^* \in \mathbb{W}$ the solution of system (3.10) is bounded [1,3]. (The proof of this statement is based on Theorem (3.1).) From here it follows that if the statements of items (i) and (ii) will be valid for arbitrary point of the set $\mathbb{W}$, then $\mathbb{W} = \mathbb{R}$ and the assertion of Theorem (3.2) is just.

Without the loss of generality it is possible to consider that by suitable replacements of the variables $y \to \alpha_1 x + y$ and $z \to \alpha_2 x + z$, the functions $f_2(x, y, z)$ and $f_3(x, y, z)$ of the right part of system (3.8) can be resulted to such form, in which $b_{11} = c_{11} = 0$. Then, in system (3.10), we will have $q_{11} \equiv 0$.

We assume that for some values of parameters system (3.8) has a periodic solution. We also suppose that $\phi(t_k) = \phi(t_0) + T \cdot k$, where $t_0 \geq 0$, a period $T \leq N \cdot \pi$ and N is positive integer; $k = 0, 1, 2, ...$

Introduce for system (3.10) the the following designations: $\xi_{ij} = s_{ij}(\phi_1(t_k))$, $\xi_{11} = p_{11}(\phi_1(t_k))$, $\xi_{12} = p_{12}(\phi_1(t_k))$, $\xi_{13} = p_{22}(\phi_1(t_k))$, $\xi_{22} = q_{12}(\phi_1(t_k))$, $\xi_{23} = q_{22}(\phi_1(t_k))$, where $i, j = 1, 2$.

Since $q_{11} \equiv 0$, then instead of system (3.10), we will consider the following system of autonomous differential equations

$$\begin{cases} \dot{x}(t) = \xi_{11}x(t) + \xi_{12}\rho(t) + \eta_{11}x^2(t) + \eta_{12}x(t)\rho(t) + \eta_{13}\rho^2(t), \\ \dot{\rho}(t) = \xi_{21}x(t) + \xi_{22}\rho(t) + \qquad\qquad\quad \eta_{22}x(t)\rho(t) + \eta_{23}\rho^2(t). \end{cases} \tag{3.11}$$

(Here system (3.11) is considering in a small neighborhood $\mathbb{O}_k$ of the point t_k: $t \in \mathbb{O}_k, k = 0, 1, 2, ...$ As initial conditions x_{k0}, ρ_{k0} for system (3.11) the solutions $x(t_k)$ and $\rho(t_k) \equiv \sqrt{x_2^2(t_k) + ... + x_n^2(t_k)}$ of system (3.8) in the point t_k are appointed.)

With the purpose of simplification of further exposition, it is possible to consider that the structure of system (3.11) (after some linear transformations) the same just as system (3.6). Thus, we have $\eta_{11} \in (-0.5, 0)$, $\eta_{12} = \eta_{21} = 0$, $\eta_{13} = 1$, $\eta_{22} = -1$, and $\eta_{23} \in (-\sqrt{2}, \sqrt{2})$.

Suppose that the time t_0 also satisfies to the condition

$$\dot{x}(t_0) = \xi_{11}x(t_0) + \xi_{12}\rho(t_0) + \eta_{11}x^2(t_0) + \rho^2(t_0) = 0.$$

By virtue of periodicity of solutions of system (3.8), we can construct the sequence t_0, t_1,..., t_k,... such that for the first equation of system (3.11) the condition $\xi_{11}x(t_k) + \xi_{12}\rho(t_k) + \eta_{11}x^2(t_k) + \rho^2(t_k) = 0$ will be fulfilled $\forall t_k$, $k = 0, 1, 2, ...$ From here it follows that

$$x(t_k) = \frac{-\xi_{11} \pm \sqrt{\xi_{11}^2 - 4\eta_{11}(\rho^2(t_k) + \xi_{12}\rho(t_k))}}{2\eta_{11}}; \; k = 0, 1, 2, ... \tag{3.12}$$

(Again by virtue of periodicity of solutions of system (3.8), we will have $\xi_{11}^2 - 4\eta_{11}(\rho^2(t_k) + \xi_{12}\rho(t_k)) > 0$.) Consequently, taking into account formula (3.12), the second equation of system (3.11) $\forall t_k$ can be transformed to the form

$$\eta_{11}\dot{\rho}^2(t_k) + C(\rho(t_k))\dot{\rho}(t_k) + \rho(t_k)F(\rho(t_k)) = 0, \tag{3.13}$$

where

$$C(\rho) = -2\eta_{11}\eta_{23}\rho^2 - (2\eta_{11}\xi_{22} + \xi_{11})\rho + \xi_{21}\xi_{11},$$

$$F(\rho) = (1+\eta_{11}\eta_{23}^2)\rho^3 + (2\eta_{11}\eta_{23}\xi_{22} + \xi_{11}\eta_{23} + \xi_{12} - 2\xi_{21})\rho^2 +$$
$$(\eta_{11}\xi_{22}^2 + \xi_{11}\xi_{22} + \xi_{21}^2 - 2\xi_{12}\xi_{21} - \xi_{21}\xi_{11}\eta_{23})\rho + (\xi_{12}\xi_{21}^2 - \xi_{21}\xi_{11}\xi_{22}).$$

From (3.13) we have

$$\dot\rho(t_k) = \frac{-C(\rho(t_k)) \pm \sqrt{C^2(\rho) - 4\eta_{11}\rho(t_k)F(\rho(t_k))}}{2\eta_{11}} =$$
$$\frac{\mp 2\rho(t_k)F(\rho(t_k))}{\sqrt{C^2(\rho(t_k)) - 4\eta_{11}\rho(t_k)F(\rho(t_k))} \pm C(\rho(t_k))} =$$
$$\frac{\mp 2\rho F(\rho(t_k))}{|\rho(t_k) - \xi_{21}|\sqrt{\xi_{11}^2 - 4\eta_{11}\rho^2(t_k) - 4\eta_{11}\xi_{12}\rho(t_k)} \pm C(\rho(t_k))}. \tag{3.14}$$

Lemma 3.1. *Let $L(\phi)$ and $D(\phi)$ be periodic nonpositive functions. Assume that the magnitude $\max_\phi |L(\phi)| \not\equiv 0$ is small enough. Then the periodic behavior of solutions of system (3.8) is generated by 1D iterated process*

$$\rho_{k+1} = \rho_k \exp\left[\frac{-2F(\rho_k)}{C(\rho_k) + |\rho_k - \xi_{21}|\sqrt{-4\eta_{11}\rho_k^2 - 4\eta_{11}\xi_{12}\rho_k + \xi_{11}^2}}\right] > 0;\ k = 0, 1, ... \tag{3.15}$$

Proof. Now we study the function

$$\Theta(\rho) = \rho \exp\left[\frac{-2F(\rho)}{C(\rho) + |\rho - \xi_{21}|\sqrt{-4\eta_{11}\rho^2 - 4\eta_{11}\xi_{12}\rho + \xi_{11}^2}}\right];\ \rho \ge 0.$$

Since $L < 0$ and $\eta_{11} < 0$, then from (3.5), we have $1 + \eta_{11}\eta_{23}^2 > 0$. Thus,

$$\lim_{\rho\to\infty} F(\rho) = \infty.$$

(b1) The function $\Theta(\rho)$ is continuous on the interval $[0, \infty)$.

Suppose the contrary. Then from (3.14) it follows that $\sqrt{C^2(\rho) - 4\eta_{11}\rho F(\rho)} + C(\rho) = 0$. Since $\eta_{11} \ne 0$, then we have either $\rho = \rho_0 = 0$ or there exists $\rho = \rho_1$ such that $F(\rho_1) = 0$ or there exists $\rho = \rho_2$ such that $C(\rho_2) = 0$.

Let $\rho = \rho_0 = 0$, $C(\rho_0) < 0$, and $F(\rho_0) > 0$. Then $\rho_0 = 0$ is a removable singularity and we have $\lim_{\rho\to 0}\Theta(\rho) = 0$. If $C(\rho_0) > 0$, then $\lim_{\rho\to 0}\Theta(\rho) = \Theta(0) = 0$.

Let $\rho_1 \ne 0$, $C(\rho_1) < 0$, and $F(\rho_1) = 0$. Then ρ_1 is a removable singularity and we have $\lim_{\rho\to 0}\Theta(\rho_1) = 0$. If $C(\rho_1) > 0$, then $\lim_{\rho\to\rho_1}\Theta(\rho) = \Theta(\rho_1) = 0$.

Let $\rho_2 \ne 0$, $C(\rho_2) = 0$, and $F(\rho_2) > 0$. Then we have $\lim_{\rho\to\rho_2}\Theta(\rho) = const \ne 0$.

In addition, since $\det A \ne 0$, we have $\xi_{11}^2 + \xi_{12}^2 \ne 0$. Further, from the condition

$$\lim_{\eta_{11}\to 0} L = \lim_{\eta_{11}\to 0} -\eta_{11}(1+\eta_{11}\eta_{23}^2) \to 0$$

and formula (3.5) it follows that $\eta_{11} \to 0$. Therefore, the quadratic function $\gamma(\rho) = -4\eta_{11}\rho^2 - 4\eta_{11}\xi_{12}\rho + \xi_{11}^2$ at $\eta_{11} \to 0$ satisfies the condition $\gamma(\rho) > 0$. Thus, $\forall \rho \geq 0$ the function $\Theta(\rho)$ is continuous on the interval $[0, \infty)$.

(b2) We rewrite equation (3.14) in such aspect

$$\dot{\rho}(t) = \frac{-2\rho F(\rho)}{C(\rho) + |\rho - \xi_{21}|\sqrt{\xi_{11}^2 - 4\eta_{11}\rho^2 - 4\eta_{11}\xi_{12}\rho}}, t \in \mathbb{O}_k, \ k = 0, 1, 2, ... \tag{3.16}$$

Let $\rho(t_k) = \rho_k$. From (3.16) it follows that

$$\rho_k = c \exp\left(\int_{t_0}^{t_k} \left[\frac{-2F(\rho(\tau))}{C(\rho(\tau)) + |\rho(\tau) - \xi_{21}|\sqrt{\xi_{11}^2 - 4\eta_{11}\rho^2(\tau) - 4\eta_{11}\xi_{12}\rho(\tau)}} \right] d\tau \right)$$

and

$$\rho_{k+1} = c \exp\left(\int_{t_0}^{t_{k+1}} \left[\frac{-2F(\rho(\tau))}{C(\rho(\tau)) + |\rho(\tau) - \xi_{21}|\sqrt{\xi_{11}^2 - 4\eta_{11}\rho^2(\tau) - 4\eta_{11}\xi_{12}\rho(\tau)}} \right] d\tau \right).$$

Having excluded from two last equalities the constant c, we get

$$\rho_{k+1} =$$
$$\rho_k \exp\left(\int_{t_k}^{t_{k+1}} \left[\frac{-2F(\rho(\tau))}{C(\rho(\tau)) + |\rho(\tau) - \xi_{21}|\sqrt{\xi_{11}^2 - 4\eta_{11}\rho^2(\tau) - 4\eta_{11}\xi_{12}\rho(\tau)}} \right] d\tau \right). \tag{3.17}$$

Further, the function $\Theta(\rho)$ is continuous on the interval $[0, \infty)$. Therefore at $\rho \to \infty$, we can represent this function in the form

$$\Theta(\rho) = \rho \exp\left[\frac{-2(1 + \eta_{11}\eta_{23}^2)\rho^3 + ...}{2\sqrt{-\eta_{11}}(1 - \sqrt{-\eta_{11}}\eta_{23})\rho^2 + ...} \right] \sim \rho \exp\left[\alpha - \frac{1 + \sqrt{-\eta_{11}}\eta_{23}}{\sqrt{-\eta_{11}}}\rho \right], \tag{3.18}$$

where $\alpha \in \mathbb{R}$. Now if we take advantage of formula (3.18), then formula (3.17) can be represented in the following form:

$$\rho_{k+1} = \rho_k \exp\left(\int_{t_k}^{t_{k+1}} \left[\alpha - \frac{1 + \sqrt{-\eta_{11}}\eta_{23}}{\sqrt{-\eta_{11}}}\rho(\tau) \right] d\tau \right). \tag{3.19}$$

Let the bounded positive function $\theta(t)$ be a monotone decreasing on interval $[t_i, t_{i+1}]$, and let it be a monotone increasing on interval $[t_{i+1}, t_{i+2}]$. Then we have (Second Theorem About Mean Value):

$$\int_{t_i}^{t_{i+2}} h(\phi(\tau)) \cdot \theta(\tau) d\tau = \int_{t_i}^{t_{i+1}} h(\phi(\tau)) \cdot \theta(\tau) d\tau + \int_{t_{i+1}}^{t_{i+2}} h(\phi(\tau)) \cdot \theta(\tau) d\tau =$$

$$= \theta(t_i + 0) \int_{t_i}^{\xi} h(\phi(\tau)) d\tau + \theta(t_{i+2} - 0) \int_{\zeta}^{t_{i+2}} h(\phi(\tau)) d\tau, \tag{3.20}$$

where $t_i \le \xi \le t_{i+1}$, $t_{i+1} \le \zeta \le t_{i+2}$. Hence, from (3.20) it follows that

$$\int_{t_i}^{t_{i+2}} h(\phi(\tau)) \cdot \theta(\tau) d\tau = p_i \theta_i + p_{i+2}\theta_{i+2}, \tag{3.21}$$

where magnitudes $p_i = \int_{t_i}^{\xi} h(\phi(\tau))d\tau, p_{i+2} = \int_{\zeta}^{t_{i+2}} h(\phi(\tau))d\tau$ can have any signs.

Now let the function $\rho(t) = \theta(t)$ be periodic. Then in (3.21), we will have $\theta_i = \theta_{i+2}$. From here it follows that

$$\rho_{i+1} = \rho_i \exp\Big[\alpha_i - \frac{1+\sqrt{-\eta_{11i}}\eta_{23i}}{\sqrt{-\eta_{11i}}}\rho_i\Big],$$

$$\rho_{i+2} = \rho_{i+1} \exp\Big[\alpha_{i+1} - \frac{1+\sqrt{-\eta_{11,i+1}}\eta_{23,i+1}}{\sqrt{-\eta_{11,i+1}}}\rho_{i+1}\Big],$$

and, therefore, we have

$$\rho_{i+2} = \rho_i \exp\Big[\alpha_i + \alpha_{i+1} - \Big(\frac{1+\sqrt{-\eta_{11,i}}\eta_{23,i}}{\sqrt{-\eta_{11,i}}} + \frac{1+\sqrt{-\eta_{11,i+1}}\eta_{23,i+1}}{\sqrt{-\eta_{11,i+1}}}\Big)\rho_i\Big]; \tag{3.22}$$

where $\forall\ i = 0, 1, ...,$ the magnitudes $\alpha = \alpha_i + \alpha_{i+1}$ and

$$\beta = \frac{1+\sqrt{-\eta_{11,i}}\eta_{23,i}}{\sqrt{-\eta_{11,i}}} + \frac{1+\sqrt{-\eta_{11,i+1}}\eta_{23,i+1}}{\sqrt{-\eta_{11,i+1}}}$$

do not depend on i (they are constants).

It is clear that at $\beta > 0$ the function $\chi(\rho) = \alpha - \beta\rho$ is decreasing on the interval $[0, \infty)$. Consequently, for $\rho \to \infty$, we have $\exp(\chi(\rho)) \to 0$. Then taking into account formula (3.22), we can derive the following formula

$$\rho_{k+2} = \rho_k \exp\Big[\alpha - \beta\rho_k\Big], k = 0, 2, 4, ...,$$

where

$$\alpha = \int_{t_k}^{t_{k+2}} \alpha(\tau)d\tau \in \mathbb{R},\ \beta = \int_{t_k}^{t_{k+2}} \frac{1+\sqrt{-\eta_{11}(\tau)}\eta_{23}(\tau)}{\sqrt{-\eta_{11}(\tau)}}d\tau > 0.$$

Now if we will take into consideration the equivalence (3.18), it is possible to get conclusion of the lemma. The proof is finished. □

It is known that condition (iii) of Theorem (3.2) is necessary in order that the solution of system (3.10) was periodic. Thus, under the conditions of Lemma (3.1) we have periodic solution of system (3.8). In this case the boundedness of solutions is obviously. □

Assume that for system (3.8) the conditions of Theorem (3.2) are not fulfilled. Then by linear nonsingular replacement of variables $(x, y, z) \to (x_1, y_1, z_1)$ it is necessary to pass from system (3.8) to a new quadratic system of differential equations. Now we apply Theorem (3.2) to the again got system.

Theorem 3.3. *Suppose that the condition (ii) of Theorem (3.2) is replaced by more weak conditions $L(\phi) \leq 0$ and $D(\phi) \leq 0$. In addition, assume that under the conditions of Theorem (3.2) the condition*

$$\liminf_{t\to\infty} \rho(t) = 0 \tag{3.23}$$

is also valid. (From this condition it follows that $\forall\epsilon > 0$ there exists a numerical sequence $t_k \to \infty$ as $k \to \infty$ such that $\rho(t_k) < \epsilon$.)

Then in system (3.8) there is a chaotic dynamic.

Proof. According to Lemma (3.1) the periodic behavior of solutions of system (3.8) is generated by 1D iterated process (3.15). In addition, the periodicity guarantees existence of solutions $t = t_k$ of the equation $\dot{x}(t) = 0$ (see the first equation of system (3.11)). It means that the magnitude $|L(\phi(t_k))| \not\equiv 0$ for such solutions is small enough.

It is well known that the function $y(\rho) = \rho \exp(\alpha - \beta\rho)$ at some values $\alpha > 0$ and $\beta > 0$ is chaotic [2, 3].

According to the results, which were derived in [2,3], in order that map (3.15) was chaotic it is necessary that for the function $\det A(\phi)$ the condition (iii) of Theorem (3.2) was satisfied . Thus, if we choose $\phi = \phi(t_0)$ such that $\det A(\phi(t_0)) < 0$, then the condition $\alpha > 0$ will be valid.

Further, from the condition (ii) of Theorem (3.2) follows that in function (3.18) the magnitude $1 + \sqrt{-\eta_{11}}\eta_{23}$ is positive. Therefore, at $\rho \to \infty$, we derive that the function $\Theta(\rho)$ is also chaotic.

Let $M = \max_{\rho\in[0,\infty)} f(\rho) > 0$ be a maximum of the function $\Theta(\rho)$. The state of chaos of the map $\Theta(\rho)$ on the interval $[0, \infty)$ can be proved by the methods offered in [2, 3].

Indeed, consider the exponential map $\rho_{k+1} = \Theta(\rho_k),\ \rho_k > 0;\ k = 0, 1, 2, ...$

Let ρ_k^* be the minimal fixed point of mapping $\Theta^{(k)}(\rho)$. It is known that for some values of parameters ξ_{ij}^*, η_{ij}^* the map $\Theta(\rho)$ is chaotic and $\lim_{k\to\infty} \rho_k^* = 0$. Then from condition (3.23) of Theorem (3.3) it follows that at the parameters ξ_{ij}^*, η_{ij}^* process (3.15) generates the subsequence $\rho_{m_1}^*, ..., \rho_{m_k}^*, ...$, for which $\rho_{m_k}^* = \lim_{t\to t_{m_k}^*} \rho(t_{m_k}^*) < \epsilon \approx 0,\ k \geq 1$. It means that the number of fixed points of mapping $\Theta^{(k)}(\rho)$ tends to ∞ as $k \to \infty$ on the finite interval $[0, M)$. In addition, the minimal fixed point tends to 0 and the maximal fixed point tends to M. From here it follows that in system (3.8) there is a chaotic dynamics. □

In the present article the situation, at which system (2.2) is a system without equilibriums (see [4]), it was not considered. It is explained by two reasons: at the study of the process Parkinson's illness, for which the model of this process did not have of equilibriums, it was absent; the methods of research of the systems without equilibriums are beyond of the present paper [4].

4. Practical applications and its analysis

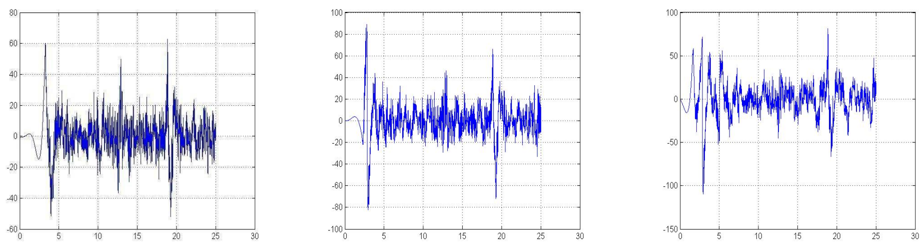
In this section we show a few practical applications of the theoretical results, which were got in previous sections. These applications are modeling the behavior of Parkinson's illness. An essence of researches consists in the following.

There are electroencephalograms of the patient, which were written in three different points of the patient cerebral cortex. We consider that these three signals represent three time series, which describe the behavior of some curve (it curve is called a disease curve) in the 3-dimensional phase space. Further, by the methods of described above, we construct the 3D system of quadratic differential equations, the solutions of which design the disease curve. (This system is determined by the coefficients of the matrix Y from Section 2.)

Now the conditions (i) and (ii) of Theorem (3.2) must be tested. In addition, it is necessary to find the values of parameters of system (3.8), at which in this system a transition from limit cycle to chaotic attractor and vice versa is take places.

In standard medical practice usually select some points on the cerebral cortex and in these points place measuring sensors. The points designate by the special characters: $Fp1,\ Fp2,\ F3,...,F8,T3,...,T8,C3,...,P3,...,O2$. In our examples we will measure electric impulses in the points $P3,P4,O1$, and $C3,C4,T5$.

We will designate the magnitudes of electric signals in points $P3,P4$ and $O1$ (see Fig.1) by coordinates $x(t)$, $y(t)$, and $z(t)$ of Cartesian coordinate system. (Similar denotations will be used in the points $C3,C4$, and $T5$.) Further, with the fixed temporal step we construct the time series x_i, y_i, and z_i in the points $P3,P4$, and $O1$ (in the points $C3,C4$, and $T5$); $i=1,...,N=5100$. In addition, we also construct the time series in the points $P3,P4$, and $O1$ in the case $i=1,...,N=3100$.



$(P3)$ $(P4)$ $(O1)$

Fig. 1. The rhythms of brain electrical activity measured in points $P3$, $P4$, and $O1$ of the cerebral cortex.

On these time series we design 3D systems of quadratic differential equations by the least square method (see Section 2):

$$\left\{\begin{aligned}
\dot{x}(t) &= -1.24 - 1.15x(t) + 2.34y(t) - 0.83z(t) - 0.04x^2(t)\\
&\qquad +0.02z(t)z(t) + 0.235x(t)y(t) - 0.015x(t)z(t) - 0.12y(t)z(t),\\
\dot{y}(t) &= +3.68 - 3.91x(t) + 1.01y(t) + 2.54z(t) - 0.16x^2(t) - 0.08y^2(t)\\
&\qquad +0.02z^2(t) + 0.15x(t)y(t) + 0.10x(t)z(t) - 0.04y(t)z(t),\\
\dot{z}(t) &= -5.9 + 5.22x(t) - 6.15y(t) - 0.31z(t) + 0.13x^2(t)\\
&\qquad +0.022y^2(t) - 0.13x(t)y(t) - 0.28x(t)z(t) + 0.05y(t)z(t);
\end{aligned}\right. \tag{4.1}$$

$$\left\{\begin{aligned}
\dot{x}(t) &= -3.11 + 0.19x(t) + 0.72y(t) - 1.19z(t) + 0.022x^2(t) - 0.04y^2(t)\\
&\qquad +0.045z^2 + 0.04x(t)y(t) - 0.01x(t)z(t) - 0.06y(t)z(t),\\
\dot{y}(t) &= -4.58 - 1.69x(t) + 0.39y(t) + 1.37z(t) - 0.03x^2(t) - 0.09y^2(t)\\
&\qquad +0.05z^2(t) + 0.11x(t)y(t) + 0.08x(t)z(t) - 0.06y(t)z(t),\\
\dot{z}(t) &= -7.88 + 6.91x(t) - 5.69y(t) - 0.71z(t) + 0.23x^2(t)\\
&\qquad +0.025y^2(t) - 0.17x(t)y(t) - 0.1x(t)z(t) + 0.06y(t)z(t);
\end{aligned}\right. \tag{4.2}$$

$$\left\{\begin{aligned}
\dot{x}(t) &= -20.93 + 1.55x(t) + 6.20y(t) - 7.05z(t) + 0.016x^2(t) + 0.17y^2(t)\\
&\qquad -0.16z^2(t) - 0.10x(t)y(t) + 0.13x(t)z(t) - 0.08y(t)z(t),\\
\dot{y}(t) &= +3.87 - 2.60x(t) + 2.12y(t) - 2.62z(t) - 0.01x^2(t) + 0.034y^2(t)\\
&\qquad -0.13z^2(t) - 0.17x(t)y(t) + 0.32x(t)z(t) + 0.025y(t)z(t),\\
\dot{z}(t) &= -12.12 + 1.36x(t) + 3.20y(t) - 3.56z(t) + 0.03x^2(t) + 0.06y^2(t)\\
&\qquad -0.14z^2(t) - 0.14x(t)y(t) + 0.09x(t)z(t) + 0.08y(t)z(t).
\end{aligned}\right. \tag{4.3}$$

Here system (4.1) (system (4.2)) simulates the impulses in points $P3$, $P4$, and $O1$ at $N = 3100$ (at $N = 5100$). System (4.3) simulates the impulses in points $C3$, $C4$, and $T5$ at $N = 5100$.

Chaotic attractors generated by systems (4.1)–(4.3) and their experimental analogues, which were built on time series, it are represented on Fig. 2–4.

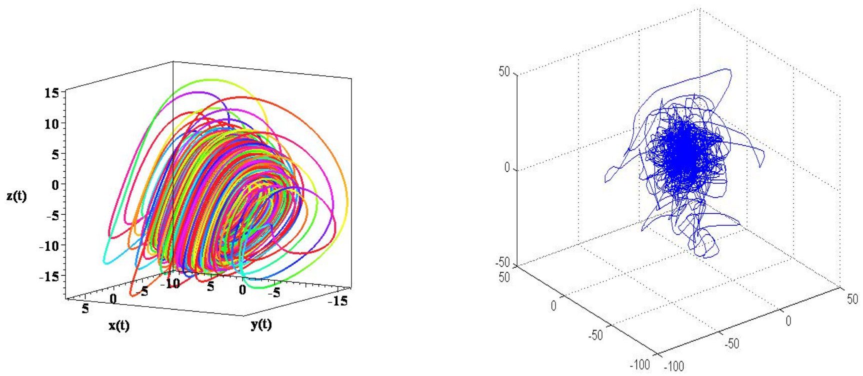


Fig. 2. The phase portraits of model (for system 4.1) and experimental (for the data set, which were represented on Fig.1) attractors measured in points $P3, P4$, and $O1$ of the cerebral cortex. The number of measurements is 3100.

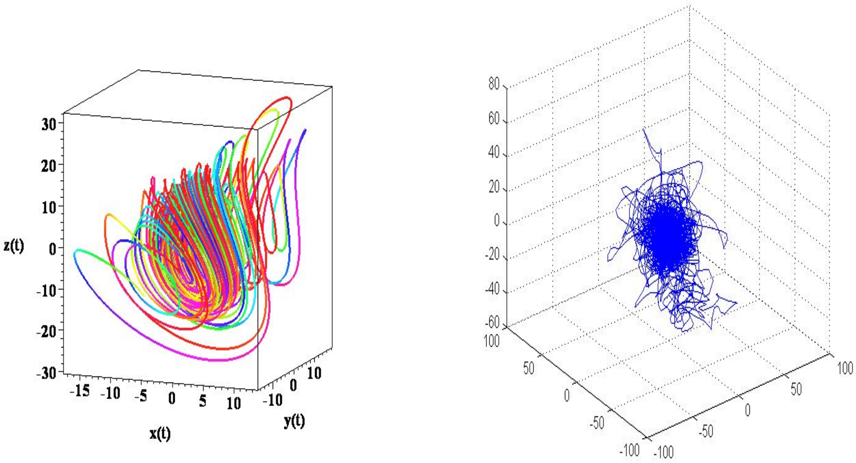


Fig. 3. The phase portraits of model (for system 4.2) and experimental (for the data set, which were represented on Fig.1) attractors measured in points $P3, P4$, and $O1$ of the cerebral cortex. The number of measurements is 5100.

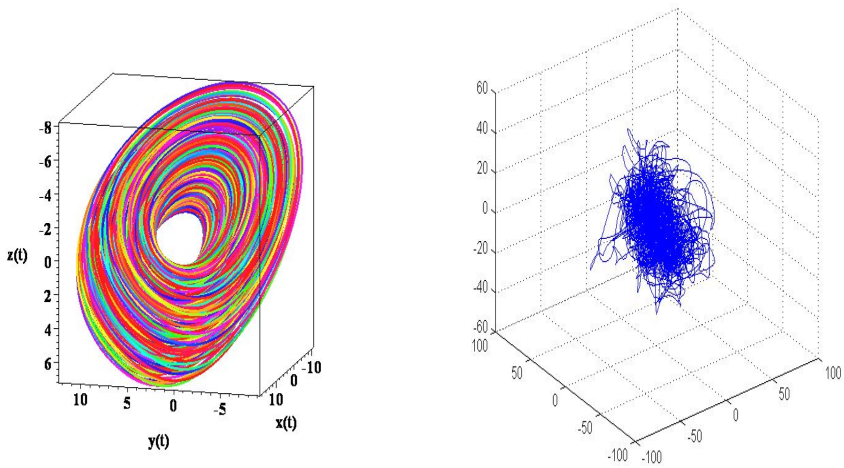


Fig. 4. The phase portraits of model (for system 4.3) and experimental attractors measured in points $C3, C4$, and $T5$ of the cerebral cortex. The number of measurements is 5100.

A partial verification of the conditions of Theorem (3.2) is shown on Fig. 5 and 6. It is necessary to notice that complete verification of all conditions of Theorems (3.2) and (3.3) is an intricate enough problem. Note that the got results show that behavior of systems (4.1), (4.2), and (4.3) generally speaking will not be chaotic. Therefore there is no necessity to check up Theorem (3.3). (It verification is desirable only in the case $L(\phi) \equiv 0$.)

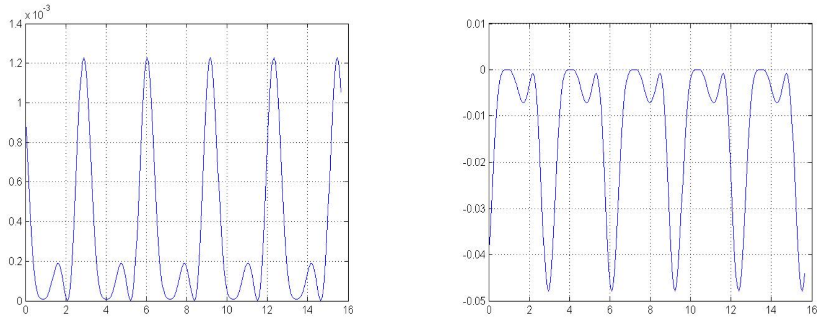


Fig. 5. The behavior of invariants $L(\phi)$ and $D(\phi)$ for system (4.2).

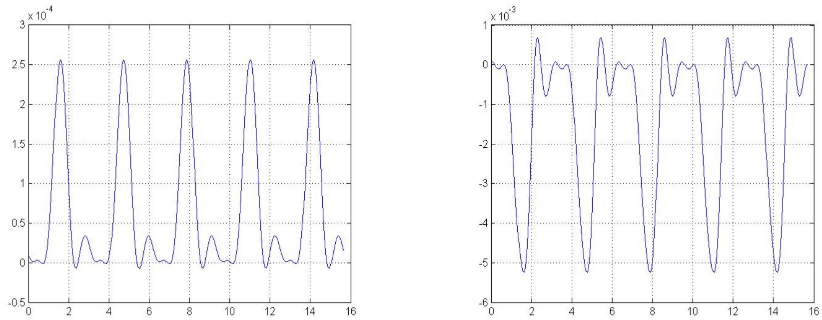


Fig. 6. The behavior of invariants $L(\phi)$ and $D(\phi)$ for system (4.3).

An information necessary for the prediction of development of illness is represented on Fig. 7 – 8.

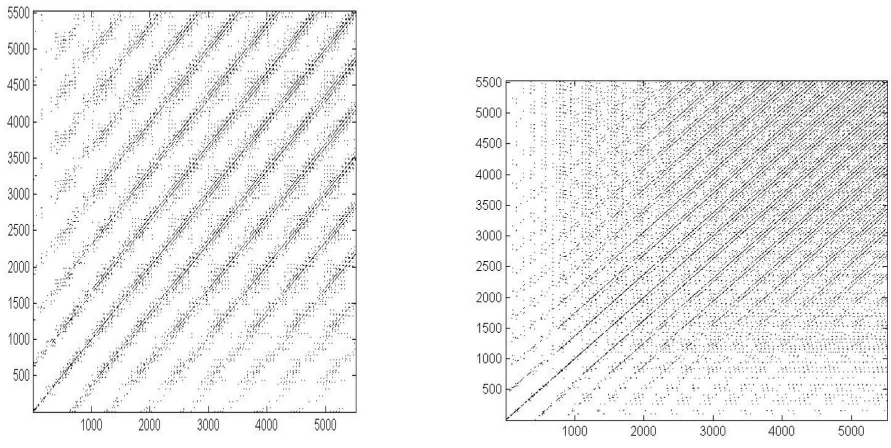
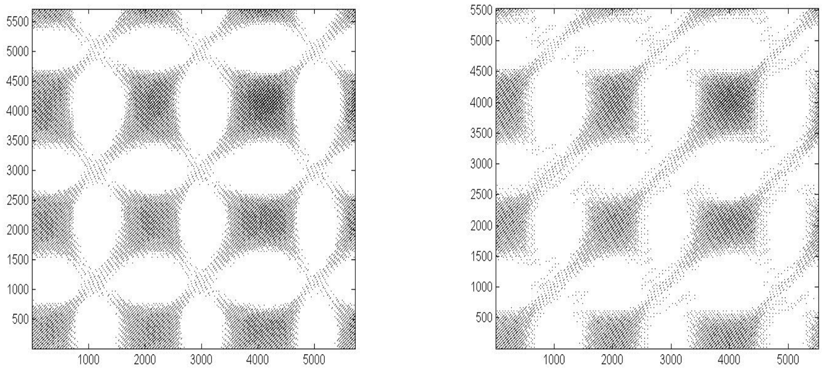


Fig. 7. The recurrence diagrams of the process of generated by the systems (4.1) and (4.2).



($\tau = 62$) ($\tau = 162$)

Fig. 8. The recurrence diagrams of the process of generated by the system (4.3) for the different time of delay τ.

The analysis of all represented results allows to do such conclusions.

1. Numerous verifications were shown that for the quadratic systems, which describe the signals of cerebrum, the invariant $L(\phi) \approx 0$ (see Fig. 5, 6). The same feature is noticed at all Lorenz-like and Chen-like systems [2, 3, 12]. Thus, the conditions of Theorem (3.2) are very restrictive for the simulation modeling. It should be said that quadratic systems, for which $L(\phi) \equiv 0$ were studied in [2–4]. A sense of the condition $L(\phi) \equiv 0$ consists in that all equations of systems (4.1) – (4.3) do not contain some quadratic summand (for example x^2 or y^2). If we will take into account this circumstance, then we can derive in systems (2.2) new models of attractors more corresponding to the experimental analogues (see Fig. 2–4).

2. Model attractors generated by systems (4.1), (4.2) differ from the attractor of system (4.3). Indeed, attractors of systems (4.1) and (4.2) are cylindrical. (It is explained those that both attractors got in points $P3, P4, O1$, but with a different number of measurements: for Fig. 2 it is $N = 3100$ and for Fig. 3 it is $N = 5100$.) The attractor of systems (4.3) is a torus. However, it is necessary to admit that the experimental attractors are rather spherical. Therefore, a further corrections of the got models are required.

3. At first sight it seems that the models of attractors of systems (4.1) and (4.2) are chaotic; the attractor of system (4.3) is quasi-periodic (see Fig. 2–4). However, as it show recurrence diagrams on Fig. 7–8, all model attractors are quasi-periodic.

Important distinctions between these diagrams consist in the following: diagrams on Fig. 8 are built for system (4.3), but with a different time of delay τ. The choice of optimum value of the parameter τ is instrumental in the increase of informative of recurrence diagram. In our case it is $\tau = 62$. The diagram on Fig. 7 shows that the periodic process passes to chaotic.

4. With respect to the medical applications of the fulfilled analysis, here it is possible to do the following conclusions. From the medical point of view complex attractor formed by the signals of cortex testifies to the normal processes flowing in this cortex [6,10]. On the contrary, the simplification of attractor and it transition to the periodic structure specifies on destruction of normal processes in a brain [6, 10]. Consequently, this destruction is the reason of disease.

On Fig. 2–4 the experimental attractors are chaotic. It means that the patient is not sick. However, the model attractors on Fig. 2–4 show that there is a progress of disease in the region $C3, C4, T5$. In region $P3, P4, O1$ the disease begins only: the chaotic mode will be replaced periodic.

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