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Проблеми математичного моделювання
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ON HÖLDER CONTINUITY OF THE COEFFICIENTS ON THE RESOLVENT OF DIRICHLET BOUNDARY VALUE PROBLEM WITH ANISOTROPIC P-LAPLACIAN

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We discuss the Hölder continuity property for the inverse mapping that identifies the diffusivity matrix $A(x)$ in the main part of anisotropic p-Laplace equation as a function of resolvent operator. In particular, we prove that, within a chosen class of non-smooth admissible matrices the resolvent determines the anisotropic diffusivity in a unique manner and the correspondent inverse mapping is Hölder continuous in suitable topologies.

Key words: anisotropic p-Laplace equations, resolvent, Hölder continuity

1. Introduction

Throughout the paper Ω is a bounded open subset of $\mathbb{R}^N$, $N \geqslant 2$, for which Poincaré's inequality holds, $p \geqslant 2$, $1/p + 1/q = 1$. Let $\mathbb{S}^N := \mathbb{R}^{\frac{N(N+1)}{2}}$ be the set of all symmetric matrices $A = [a_{ij}]_{i,j=1}^N$, ($a_{ij} = a_{ji} \in \mathbb{R}$). We suppose that $\mathbb{S}^N$ is endowed with the Euclidian scalar product $A \cdot B = \mathrm{tr}(A\,B) = a_{ij}b_{ij}$ and with the corresponding Euclidian norm $\|A\|_{\mathbb{S}^N} = (A \cdot A)^{1/2}$. We also make use of the so-called spectral norm $\|A\|_2 := \sup\left\{|A\xi| \ : \ \xi \in \mathbb{R}^N \ \text{ with } \ |\xi| = 1\right\}$ of matrices $A \in \mathbb{S}^N$, which is different from the Euclidean norm $\|A\|_{\mathbb{S}^N}$. However, the relation $\|A\|_2 \leqslant \|A\|_{\mathbb{S}^N} \leqslant \sqrt{N}\|A\|_2$ holds true for all $A \in \mathbb{S}^N$.

Admissible matrices Let α and β be given distributions from $L^\infty(\Omega)$ such that

$$\alpha > 0 \ \text{ a.e. in } \Omega, \quad \alpha^2 \leqslant \beta \ \text{ a.e. in } \Omega, \text{ and } \quad \alpha^{-q} \in L^1(\Omega). \tag{1.1}$$

Let us define the following class of matrices

$$\mathcal{M}(\Omega) = \left\{A(x) \in L^\infty(\Omega;\mathbb{S}^N) : \ \alpha^2(x)I \leqslant A(x) \leqslant \beta(x)I \text{ a.e. in } \Omega\right\}, \tag{1.2}$$

where inequalities in (1.2) should be considered in the sense of the corresponding quadratic forms. The norm of $A(x)$ we further define as a spectral one $\|A(x)\|_2$. We say that $A = [a_{ij}]_{i,j=1}^N$ is a matrix of anisotropic diffusivity if $A \in \mathcal{M}(\Omega)$.

In this paper we derive some sensitivity estimates for the solutions to the following boundary value problem for the degenerate quasi-linear elliptic equation (in what follows we shall call (1.3) the anisotropic p-Laplace equation)

$$-\mathrm{div}\big(|(A\nabla y, \nabla y)_{\mathbb{R}^N}|^{\frac{p-2}{2}} A\nabla y\big) = f \text{ in } \Omega, \tag{1.3}$$

$$y = 0 \text{ on } \partial\Omega, \tag{1.4}$$

where $A \in \mathcal{M}(\Omega)$, $f = -\mathrm{div} g$, $g \in L^\infty(\Omega)$ is a given distribution.

Our main goal is to analyze the inverse problem of identifying the matrix of anisotropic diffusivity $A(x)$ in the principle part of quasi-linear elliptic equation (1.3) as a function of resolvent operator. In particular, we prove that, within the class of admissible matrices $\mathcal{M}(\Omega)$, the resolvent determines the anisotropic diffusivity in a unique manner. Furthermore we prove that the inverse mapping from resolvent to the matrix A is Hölder continuous in suitable topologies.

Our main results can be stated as follows.

Theorem 1.1. *Let A and B be given elements from $\mathcal{M}(\Omega)$ such that*

$$^*[B^{-\frac{1}{2}}]AB^{-\frac{1}{2}} \in L^\infty(\Omega;\mathbb{S}^N) \text{ and } \left\| ^*[B^{-\frac{1}{2}}]AB^{-\frac{1}{2}} \right\|^{\frac{p}{2}}_{L^\infty(\Omega;\mathbb{S}^N)} \leqslant 1.$$

Let $f = -\mathrm{div} g$ for some $g \in L^\infty(\Omega;\mathbb{R}^N)$. Then

$$1 - \left\| ^*[B^{-\frac{1}{2}}]AB^{-\frac{1}{2}} \right\|^{\frac{p}{2}}_{L^\infty(\Omega;\mathbb{S}^N)}$$
$$\leqslant \begin{cases} \|\mathcal{R}_B(f) - \mathcal{R}_A(f)\|_{H^2_A(\Omega)}, & \text{for } p = 2, \\ \left[1 + \|\mathcal{R}_B(f) + \mathcal{R}_A(f)\|^{\frac{p-2}{2}}_{H^p_A(\Omega)} \|\mathcal{R}_A(f)\|_{H^p_A(\Omega)}\right] \|\mathcal{R}_B(f) - \mathcal{R}_A(f)\|_{H^p_A(\Omega)}, & \text{for } p \in (2;4], \\ \left[1 + \dfrac{p-2}{2}\left(\|\mathcal{R}_A(f)\|_{H^p_A(\Omega)} + 1\right)^{p-2}\right] \|\mathcal{R}_B(f) - \mathcal{R}_A(f)\|_{H^p_A(\Omega)}, & \text{for } p > 4 \end{cases}$$

where the weighted Sobolev space $H^p_A(\Omega)$ is defined in (3.1), by R_A we denote the inverse or resolvent operator for problem (1.3)–(1.4), which is uniquely determined by the matrix $A \in \mathcal{M}(\Omega)$ and maps continuously $\big(H^p_A(\Omega)\big)^*$ into $H^p_A(\Omega)$, while the value $1 - \left\| ^*[B^{-\frac{1}{2}}]AB^{-\frac{1}{2}} \right\|^{\frac{p}{2}}_{L^\infty(\Omega;\mathbb{S}^N)}$ one can interpret as a measure of closedness between matrices A and B (see Remark 4.2).

Theorem 1.2. *Let A, $B \in (M)(\Omega)$ be given matrices such that*

$$^*[B^{-\frac{1}{2}}]AB^{-\frac{1}{2}},\ ^*[A^{-\frac{1}{2}}](A-B)B^{-\frac{1}{2}} \in L^\infty(\Omega;\mathbb{S}^N).$$

Then for $p \geqslant 2$ *and* $f = -\mathrm{div} g$, $g \in L^\infty(\Omega;\mathbb{R}^N)$, *we have*

$$\begin{aligned} &C_p\|\mathcal{R}_B(f)-\mathcal{R}_A(f)\|^{p-1}_{H^p_A(\Omega)} \\ &\leqslant C(A,B,g,\alpha^{-1})\begin{cases} \|^*[B^{-\frac{1}{2}}](A-B)B^{-\frac{1}{2}}\|^{\frac{p-2}{2}}_{L^\infty(\Omega;\mathbb{S}^N)}, & p\in[2,4],\\ \|^*[B^{-\frac{1}{2}}](A-B)B^{-\frac{1}{2}}\|_{L^\infty(\Omega;\mathbb{S}^N)}, & p>4, \end{cases}\end{aligned}$$

for some constant $C = C(A,B,g,\alpha^{-1})$ *(see Lemma 3.1 for the details of its identification).*

The results under discussion are close to the questions of stability and sensitivity analysis for boundary value problems and optimization problems associated with them. Stability refers to the continuous behavior of solutions under small perturbations of the problem data, while sensitivity indicates a differentiable dependence (see, [8]). It is worth to notice that, by analogy with a linear case [9], this result plays a key role when applying greedy algorithms to the approximation of parameter-dependent quasi-linear elliptic problems with anisotropic p-laplacian in an uniform and robust manner, independent of the given source terms (see, for instance, [2–4]).

2. Physical Motivation

To begin with we note that equation (1.3) can be viewed as the Euler equation for the variational integral

$$J(u)=\frac{1}{p}\int_\Omega |(A\nabla u,\nabla u)_{\mathbb{R}^N}|^{\frac{p}{2}}\,dx-\int_\Omega (g,\nabla u)_{\mathbb{R}^N}\,dx\to\inf \qquad (2.1)$$

and its interest arises from various applied context related to composite materials (such as nonlinear dielectric composites), whose nonlinear behavior is modeled by the so-called power law.

Another application of the energy functional (2.1) and, therefore, equation (1.3), can be found in the shape optimization theory. Indeed, let D be a given nonempty domain in $\mathbb{R}^N$, and let

$$\mathcal{P}(D)=\{\Omega\,:\,\Omega\subset D,\ \Omega \text{ is open}\}$$

be the set of all open subsets of D. Let $V:[0,\tau]\times\mathbb{R}^N\to\mathbb{R}^N$ be a given velocity field. Consider the transformations $\{T_t:\mathbb{R}^N\to\mathbb{R}^N,\ t\in[0,\tau]\}$ defined as

$$x\to T_t(x):=y(t,x),$$

where $y(t,x)$ is a solution (flow) of the differential equation

$$\begin{aligned}\frac{d\,y(t,x)}{dt}&=V(t,y(t,x)),\ t\geqslant 0,\\ y(0,x)&=x.\end{aligned}$$

Then, for a given domain $\Omega \subset D$, we can associate with $t > 0$ the new set (certain perturbation of Ω)

$$\Omega_t := T_t(\Omega) = \{T_t(x),\ \forall\, x \in \Omega\},$$

i.e. some transport of the set Ω by the velocity field V. This type of perturbations of the initial domain Ω plays a crucial role in the study of shape optimization problems.

So, if we assume that the p-Laplace equation

$$-\mathrm{div}\left(|\nabla\, y|^{p-2}\nabla\, y\right) = f \text{ in } W_0^{1,p}(\Omega) \tag{2.2}$$

is defined on a given bounded Lipschitz domain Ω and the associated energy is given by

$$J(0,\varphi) = \frac{1}{p}\int_\Omega |\nabla\, \varphi|^p\, dx - \int_\Omega f\, \varphi\, dx,\ \forall\, \varphi \in W_0^{1,p}(\Omega),$$

then the energy on a perturbed domain Ω_t can be expressed as follows (see Delfour, Zolessio for the details)

$$\begin{aligned}\widetilde{J}(t,\varphi) &= \frac{1}{p}\int_{\Omega_t = T_t(\Omega)} |\nabla\, \varphi|^p\, dx - \int_{T_t(\Omega)} f\, \varphi\, dx\\ &= \frac{1}{p}\int_\Omega \lambda_t |{}^*[\mathcal{D}T_t]^{-1}\nabla\, \widetilde{\varphi}|^p\, dx - \int_\Omega f\, \lambda_t \widetilde{\varphi}\, dx,\end{aligned} \tag{2.3}$$

where $\mathcal{D}T_t$ is the Jacobian matrix of T_t, ${}^*[\mathcal{D}T_t]^{-1}$ is the transpose of $[\mathcal{D}T_t]^{-1}$, $\lambda_t = |\mathrm{det}\mathcal{D}T_t|$, and $\widetilde{\varphi} = \varphi \circ T_t$. Hence, the minimization of $\widetilde{J}(t,\cdot)$ over $W_0^{1,p}(T_t(\Omega))$ is equivalent to the minimization of $J(t,\varphi) = \widetilde{J}(t, \widetilde{\varphi} \circ T_t^{-1})$ over $W_0^{1,p}(\Omega)$.

As a result, the corresponding Euler equation for (2.3) takes the form

$$\begin{gathered}-\mathrm{div}\left(|G(t)\nabla\, y^t|^{p-2}\, {}^*[G(t)]G(y)\nabla\, y^t\right) = \lambda_t\, f \circ T_t \text{ in } \Omega,\\ y^t = 0 \text{ on } \partial\Omega,\end{gathered}$$

where $G(t) := \sqrt[p]{\lambda_t}\, {}^*[\mathcal{D}T_t]^{-1}$. Thus, having put $A = {}^*[G]G$, we arrive at the anisotropic p-Laplace equation (1.3)–(1.4).

3. Preliminaries and Auxiliary Results

Weighted Sobolev spaces. In order to furnish the boundary value problem (1.3)–(1.4) by some functional space description, we associate with each matrix $A \in \mathcal{M}(\Omega)$ the weighted Sobolev space

$$H_A^p(\Omega) = W_0^{1,p}(\Omega; A\, dx),$$

which we define as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|y\|_A = \left(\int_\Omega |y|^p\, dx + \int_\Omega |(A\nabla\, y, \nabla\, y)_{\mathbb{R}^N}|^{\frac{p}{2}} dx\right)^{\frac{1}{p}}. \tag{3.1}$$

Note that, due to inequalities (1.2) and estimates

$$\begin{aligned}
\int_\Omega |y|\,dx &\leqslant \left(\int_\Omega |y|^p\right)^{1/p} |\Omega|^{1/q} \leqslant C\|y\|_A,\\
\int_\Omega |\nabla\, y|\,dx &\leqslant \left(\int_\Omega |\nabla y|^p \alpha^p dx\right)^{1/p} \left(\int_\Omega \alpha^{-q}\,dx\right)^{1/q}\\
&\leqslant \|\alpha^{-q}\|^{1/q}_{L^1(\Omega)} \left(\int_\Omega |A^{\frac{1}{2}}\nabla\, y|^p\,dx\right)^{1/p}\\
&= C\left(\int_\Omega |(A\nabla\, y, \nabla\, y)_{\mathbb{R}^N}|^{p/2} dx\right)^{1/p} \leqslant C\|y\|_A,
\end{aligned}$$

the space $H^p_A(\Omega)$ is complete with respect to the norm $\|\cdot\|_A$ (see [1]). Moreover, following [6], we have the following result: if there exists a real $\nu \in \left(\frac{N}{p}, +\infty\right) \cap \left[\frac{1}{p-1}, +\infty\right)$ such that $\alpha^{-\nu p} \in L^1(\Omega)$, then the expression

$$\|y\|_{H^p_A(\Omega)} = \left(\int_\Omega |(A(x)\nabla\, y, \nabla\, y)_{\mathbb{R}^N}|^{p/2}\,dx\right)^{1/p}$$

can be considered as a norm on $H^p_A(\Omega)$ and it is equivalent to the norm (3.1). Besides, in this case the embedding $H^p_A(\Omega) \hookrightarrow L^p(\Omega)$ is compact.

Hereinafter we denote by $\langle\cdot,\cdot\rangle_{-1;1}$ the duality paring between $\left(H^p_A(\Omega)\right)^*$ and $H^p_A(\Omega)$.

Anisotropic p-Laplacian. Before proceeding further, we indicate some well-known properties of the operator

$$\mathcal{A}_A(y) = -\mathrm{div}\left(|(A\nabla y, \nabla y)_{\mathbb{R}^N}|^{\frac{p-2}{2}} A\nabla y\right).$$

1. For every $A \in \mathcal{M}(\Omega)$ the operator $\mathcal{A}_A$ acts from $H^p_A(\Omega)$ to $\left(H^p_A(\Omega)\right)^*$ and

$$\begin{aligned}
\langle \mathcal{A}_A(y), v\rangle_{-1;1} &= \int_\Omega |(A\nabla y, \nabla y)_{\mathbb{R}^N}|^{\frac{p-2}{2}} (A\nabla y, \nabla\, v)_{\mathbb{R}^N}\,dx\\
&= \int_\Omega |A^{\frac{1}{2}}\nabla y|^{p-2} (A^{\frac{1}{2}}\nabla y, A^{\frac{1}{2}}\nabla\, v)_{\mathbb{R}^N}\,dx\\
&\leqslant \left(\int_\Omega |A^{\frac{1}{2}}\nabla y|^p\right)^{\frac{p-1}{p}} \left(\int_\Omega |A^{\frac{1}{2}}\nabla v|^p\right)^{\frac{1}{p}}\\
&= \|y\|^{p-1}_{H^p_A(\Omega)} \|v\|_{H^p_A(\Omega)}.
\end{aligned} \tag{3.2}$$

2. The function $\mathbb{R} \ni t \to \langle \mathcal{A}(y + tv), w\rangle_{-1;1}$ is continuous for all $y, v, w \in H^p_A(\Omega)$, i.e. semi-continuity property holds for $\mathcal{A}_A$.

3. Operator $\mathcal{A}_A$ is strictly monotone. Indeed, this property follows from the well-known estimates

$$(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta)(\xi-\eta)\geqslant\begin{cases}C_p|\xi-\eta|^p, & p\geqslant 2,\\ C_p\left(|\xi|+|\eta|\right)^{p-2}|\xi-\eta|^2, & 1<p\leqslant 2,\end{cases}\tag{3.3}$$

for all $\xi,\ \eta\in\mathbb{R}^N$, where the constant $C_p=2^{2-p}$ for $p\geqslant 2$ and $C_p=1$ for $1<p<2$.

4. The operator $\mathcal{A}_A$ is coercive in the following sense

$$\lim_{\|y\|_{H^p_A(\Omega)}\to\infty}\frac{\langle\mathcal{A}_A(y),y\rangle_{-1;1}}{\|y\|_{H^p_A(\Omega)}}=+\infty.\tag{3.4}$$

Indeed, following the definition of $\mathcal{A}_A$, we have

$$\begin{aligned}\langle\mathcal{A}_A(y),y\rangle_{-1;1}&=\int_\Omega|(A\nabla y,\nabla y)_{\mathbb{R}^N}|^{\frac{p-2}{2}}(A\nabla y,\nabla\, y)_{\mathbb{R}^N}\,dx\\&=\int_\Omega|A^{\frac12}\nabla y|^p\,dx=\|y\|^p_{H^p_A(\Omega)},\end{aligned}$$

therefore (3.4) becomes obvious.

Let $f\in\left(H^p_A(\Omega)\right)^*$ be a given distribution. We consider the following variational problem

$$\begin{cases}\text{Find } y\in H^p_A(\Omega)\text{ such that}\\ \displaystyle\int_\Omega|(A\nabla y,\nabla y)_{\mathbb{R}^N}|^{\frac{p-2}{2}}(A\nabla y,\nabla\,\varphi)_{\mathbb{R}^N}\,dx=\langle f,\varphi\rangle_{-1;1},\ \forall\,\varphi\in C^\infty_0(\Omega).\end{cases}\tag{3.5}$$

According to the well-known theorem on nonlinear operator equations with monotone operators, (3.5) has a unique solution $y\in H^p_A(\Omega)$ for every $A\in\mathcal{M}(\Omega)$. Moreover, the energy equality

$$\int_\Omega|A^{\frac12}\nabla y|^p\,dx=\langle f,y\rangle_{-1;1}$$

implies

$$\int_\Omega\alpha^p|\nabla\,y|^p\,dx\leqslant\int_\Omega|A^{\frac12}\nabla y|^p\,dx\leqslant\|f\|_{\left(H^p_A(\Omega)\right)^*}\|y\|_{H^p_A(\Omega)}.$$

Hence,

$$\|y\|_{H^p_A(\Omega)}\leqslant\|f\|^{1/(p-1)}_{\left(H^p_A(\Omega)\right)^*}.\tag{3.6}$$

Thus, the matrix $A\in\mathcal{M}(\Omega)$ determines uniquely the inverse or resolvent operator R_A, which maps continuously $\left(H^p_A(\Omega)\right)^*$ into $H^p_A(\Omega)$. We address the inverse problem consisting on identifying the matrix A in terms of the resolvent R_A.

We begin with the following result.

Lemma 3.1. *Let A, $B \in (M)(\Omega)$ be given matrices such that*

$$^*[B^{-\frac{1}{2}}]AB^{-\frac{1}{2}},\ ^*[A^{-\frac{1}{2}}](A-B)B^{-\frac{1}{2}} \in L^\infty(\Omega;\mathbb{S}^N).$$

Then for $p \geqslant 2$ and $f = -\mathrm{div} g$, $g \in L^\infty(\Omega;\mathbb{R}^N)$ we have

$$C_p \|y_A - y_B\|^{p-1}_{H^p_A(\Omega)} \leqslant C(A,B,g,\alpha^{-1}) \begin{cases} \|^*[B^{-\frac{1}{2}}](A-B)B^{-\frac{1}{2}}\|^{\frac{p-2}{2}}_{L^\infty(\Omega;\mathbb{S}^N)}, & p \in [2,4], \\ \|^*[B^{-\frac{1}{2}}](A-B)B^{-\frac{1}{2}}\|_{L^\infty(\Omega;\mathbb{S}^N)}, & p > 4, \end{cases}$$

where for the each case the constant $C = C(A,B,g,\alpha^{-1})$ will be identified later on.

Proof. From (3.5) we have

$$\int_\Omega |(A\nabla y_A, \nabla y_A)_{\mathbb{R}^N}|^{\frac{p-2}{2}} (A\nabla y_A, \nabla\,\varphi)_{\mathbb{R}^N}\,dx = \int_\Omega |(B\nabla y_B, \nabla y_B)_{\mathbb{R}^N}|^{\frac{p-2}{2}} (B\nabla y_B, \nabla\,\varphi)_{\mathbb{R}^N}\,dx,\ \forall\,\varphi \in C^\infty_0(\Omega), \quad (3.7)$$

where $y_A = R_A(f)$ and $y_B = R_B(f)$.

Since $H^p_A(\Omega) \subset W^{1,1}_0(\Omega)$, $H^p_B(\Omega) \subset W^{1,1}_0(\Omega)$, and $C^\infty_0(\Omega)$ is dense in $W^{1,1}_0(\Omega)$, it follows that integral identity (3.7) can be extended (by continuity) to the particular choice of $\varphi = y_A - y_B$. As a result, we deduce from (3.7) the following relation

$$\begin{aligned} &\int_\Omega \Big(|A^{\frac{1}{2}}\nabla y_A|^{p-2} A\nabla y_A - |A^{\frac{1}{2}}\nabla y_B|^{p-2} A\nabla\, y_B, \nabla\, y_A - \nabla\, y_B\Big)_{\mathbb{R}^N} dx \\ &\quad + \int_\Omega |A^{\frac{1}{2}}\nabla y_B|^{p-2} (A\nabla\, y_B - B\,\nabla y_B, \nabla\, y_A - \nabla\, y_B)_{\mathbb{R}^N}\,dx \\ &\quad + \int_\Omega \Big(|A^{\frac{1}{2}}\nabla y_B|^{p-2} - |B^{\frac{1}{2}}\nabla y_B|^{p-2}\Big) (B\nabla\, y_B, \nabla\, y_A - \nabla\, y_B)_{\mathbb{R}^N}\,dx \\ &= I_1 + I_2 + I_3 = 0. \end{aligned} \quad (3.8)$$

Then, the strict monotonicity property (3.4) and the fact that $A = {}^*[A^{\frac{1}{2}}]A^{\frac{1}{2}}$, we have

$$\begin{aligned} I_1 &= \int_\Omega \Big(|A^{\frac{1}{2}}\nabla y_A|^{p-2} A^{\frac{1}{2}}\nabla y_A - |A^{\frac{1}{2}}\nabla y_B|^{p-2} A^{\frac{1}{2}}\nabla\, y_B, \nabla\, A^{\frac{1}{2}} y_A - A^{\frac{1}{2}}\nabla\, y_B\Big)_{\mathbb{R}^N} dx \\ &\geqslant C_p \int_\Omega |A^{\frac{1}{2}}\nabla\, y_A - A^{\frac{1}{2}}\nabla\, y_B|^p dx = C_p \|y_A - y_B\|^p_{H^p_A(\Omega)}. \end{aligned} \quad (3.9)$$

In order to estimate the term I_2 we make use of the Hölder inequality

$$\int_\Omega f_1\, f_2\, f_3\,dx \leqslant \left(\int_\Omega |f_1|^{p_1} dx\right)^{1/p_1} \left(\int_\Omega |f_2|^{p_2} dx\right)^{1/p_2} \left(\int_\Omega |f_3|^{p_3} dx\right)^{1/p_3} \quad (3.10)$$

with $\dfrac{1}{p_1}+\dfrac{1}{p_2}+\dfrac{1}{p_3}=1$, $p_i>1$, $i=1,2,3$. Having set

$$p_1=\frac{p}{p-2}, \qquad p_2=p_3=p,$$

we get

$$\begin{aligned}
|I_2| &\leqslant \int_\Omega |A^{\frac12}\nabla y_B|^{p-2}\left|\left({}^*[A^{-\frac12}](A-B)B^{-\frac12}B^{\frac12}\nabla y_B, A^{\frac12}\nabla y_A - A^{\frac12}\nabla y_B\right)_{\mathbb{R}^N}\right| dx\\
&\leqslant \underbrace{\|{}^*[A^{-\frac12}](A-B)B^{-\frac12}\|_{L^\infty(\Omega;\mathbb{S}^N)}}_{C(A,B)}\\
&\qquad \times \int_\Omega |A^{\frac12}\nabla y_B|^{p-2}|B^{\frac12}\nabla y_B||A^{\frac12}\nabla y_A - A^{\frac12}\nabla y_B| dx\\
&\overset{\text{by (3.10)}}{\leqslant} C(A,B)\,\|y_B\|^{p-2}_{H^p_A(\Omega)}\|y_B\|_{H^p_B(\Omega)}\|y_A-y_B\|_{H^p_A(\Omega)}\\
&\overset{\text{by (3.6)}}{\leqslant} C(A,B)\,\|f\|^{\frac{1}{p-1}}_{\left(H^p_B(\Omega)\right)^*}\,\|y_B\|^{p-2}_{H^p_A(\Omega)}\|y_A-y_B\|_{H^p_A(\Omega)}.
\end{aligned} \tag{3.11}$$

Since

$$\begin{aligned}
\langle f,\varphi\rangle_{\left(H^p_B(\Omega)\right)^*;H^p_B(\Omega)} &= \int_\Omega (g,\nabla\varphi)_{\mathbb{R}^N}dx = \int_\Omega \left({}^*[B^{-\frac12}]g, B^{\frac12}\nabla\varphi\right)_{\mathbb{R}^N} dx\\
&\leqslant \left(\int_\Omega |B^{-\frac12}g|^q dx\right)^{1/q}\left(\int_\Omega |(B\nabla\varphi,\nabla\varphi)_{\mathbb{R}^N}|^{\frac{p}{2}}dx\right)^{1/p}\\
&= \left(\int_\Omega |(B^{-1}g,g)_{\mathbb{R}^N}|^{\frac{q}{2}}dx\right)^{1/q}\|\varphi\|_{H^p_B(\Omega)} \leqslant \left(\int_\Omega \alpha^{-q}|g|^q dx\right)^{1/q}\|\varphi\|_{H^p_B(\Omega)}\\
&\leqslant \|g\|_{L^\infty(\Omega;\mathbb{R}^N)}\|\alpha^{-1}\|_{L^q(\Omega)}\|\varphi\|_{H^p_B(\Omega)},\ \forall\,\varphi\in C_0^\infty(\Omega),
\end{aligned}$$

it follows that

$$\|f\|_{\left(H^p_B(\Omega)\right)^*} \leqslant \|g\|_{L^\infty(\Omega;\mathbb{R}^N)}\|\alpha^{-1}\|_{L^q(\Omega)}, \quad \forall\, B\in\mathcal{M}(\Omega). \tag{3.12}$$

Besides, we note that

$$\begin{aligned}
\|y_B\|^p_{H^p_A(\Omega)} &= \int_\Omega |(A\nabla y_B,\nabla y_B)_{\mathbb{R}^N}|^{\frac{p}{2}}\,dx\\
&= \int_\Omega |({}^*[B^{-\frac12}]AB^{-\frac12}B^{\frac12}\nabla y_B, B^{\frac12}\nabla y_B)_{\mathbb{R}^N}|^{\frac{p}{2}}dx\\
&\leqslant \int_\Omega \left|{}^*[B^{-\frac12}]AB^{-\frac12}\right|^{\frac{p}{2}}_{\mathbb{S}^N}|B^{\frac12}\nabla y_B|^p dx\\
&\leqslant \|{}^*[B^{-\frac12}]AB^{-\frac12}\|^{\frac{p}{2}}_{L^\infty(\Omega;\mathbb{S}^N)}\|y_B\|^p_{H^p_B(\Omega)}\\
&\overset{\text{by (3.6)}}{\leqslant} \|{}^*[B^{-\frac12}]AB^{-\frac12}\|^{\frac{p}{2}}_{L^\infty(\Omega;\mathbb{S}^N)}\|f\|^{\frac{p}{p-1}}_{\left(H^p_B(\Omega)\right)^*}\\
&\overset{\text{by (3.12)}}{\leqslant} \|{}^*[B^{-\frac12}]AB^{-\frac12}\|^{\frac{p}{2}}_{L^\infty(\Omega;\mathbb{S}^N)}\|g\|^q_{L^\infty(\Omega;\mathbb{R}^N)}\|\alpha^{-1}\|^q_{L^q(\Omega)}.
\end{aligned} \tag{3.13}$$

Combining estimates (3.12)–(3.13) with (3.11), we obtain

$$\begin{aligned}
|I_2| &\leqslant \|{}^*[A^{-\frac{1}{2}}](A-B)B^{-\frac{1}{2}}\|_{L^\infty(\Omega;\mathbb{S}^N)}\|g\|^{\frac{1}{p-1}}_{L^\infty(\Omega;\mathbb{R}^N)}\|\alpha^{-1}\|^{\frac{1}{p-1}}_{L^q(\Omega)}\\
&\quad\times \|{}^*[B^{-\frac{1}{2}}]AB^{-\frac{1}{2}}\|^{\frac{p-2}{2}}_{L^\infty(\Omega;\mathbb{S}^N)}\|g\|^{\frac{p-2}{p-1}}_{L^\infty(\Omega;\mathbb{R}^N)}\\
&\quad\times \|\alpha^{-1}\|^{\frac{p-2}{p-1}}_{L^q(\Omega)} \times \|y_A - y_B\|_{H^p_A(\Omega)}\\
&= \|{}^*[A^{-\frac{1}{2}}](A-B)B^{-\frac{1}{2}}\|_{L^\infty(\Omega;\mathbb{S}^N)}\|{}^*[B^{-\frac{1}{2}}]AB^{-\frac{1}{2}}\|^{\frac{p-2}{2}}_{L^\infty(\Omega;\mathbb{S}^N)}\\
&\quad\times \|g\|_{L^\infty(\Omega;\mathbb{R}^N)}\|\alpha^{-1}\|_{L^q(\Omega)}\|y_A-y_B\|_{H^p_A(\Omega)}, \qquad (3.14)
\end{aligned}$$

where

$$\begin{aligned}
{}^*[A^{-\frac{1}{2}}](A-B)B^{-\frac{1}{2}} &= {}^*[A^{-\frac{1}{2}}]({}^*[A^{\frac{1}{2}}]A^{\frac{1}{2}} - {}^*[B^{\frac{1}{2}}]B^{\frac{1}{2}})B^{-\frac{1}{2}}\\
&= A^{\frac{1}{2}}B^{-\frac{1}{2}} - {}^*[B^{\frac{1}{2}}A^{-\frac{1}{2}}], \qquad (3.15)\\
{}^*[A^{-\frac{1}{2}}](A-B)B^{-\frac{1}{2}} &= {}^*[A^{-\frac{1}{2}}]{}^*[B^{\frac{1}{2}}]{}^*[B^{-\frac{1}{2}}](A-B)B^{-\frac{1}{2}}\\
&= {}^*[B^{\frac{1}{2}}A^{-\frac{1}{2}}]{}^*[B^{-\frac{1}{2}}](A-B)B^{-\frac{1}{2}} \qquad (3.16)
\end{aligned}$$

In order to evaluate the last term I_3, we make use of the following two inequalities:

$$\left||\xi|^{\frac{p-2}{2}} - |\eta|^{\frac{p-2}{2}}\right| \leqslant \begin{cases} \frac{p-2}{2}\left(|\xi|+|\eta|\right)^{\frac{p-4}{2}}|\xi-\eta|, & p>4, \quad \forall\,\xi,\ \eta\in\mathbb{R}^N,\\ |\xi-\eta|^{\frac{p-2}{2}}, & 2\leqslant p\leqslant 4, \quad \forall\,\xi,\ \eta\in\mathbb{R}^N, \end{cases} \qquad (3.17)$$

Taking these into account, we get for $2 \leqslant p \leqslant 4$

$$\begin{aligned}
|I_3| &\leqslant \int_\Omega \left||(A\nabla y_B,\nabla y_B)_{\mathbb{R}^N}|^{\frac{p-2}{2}} - |(B\nabla y_B,\nabla y_B)_{\mathbb{R}^N}|^{\frac{p-2}{2}}\right| |{}^*[A^{-\frac{1}{2}}]B\nabla y_B|\\
&\quad\times |A^{\frac{1}{2}}\nabla y_A - A^{\frac{1}{2}}\nabla y_B|\,dx\\
&\leqslant \int_\Omega |((A-B)\nabla y_B,\nabla y_B)_{\mathbb{R}^N}|^{\frac{p-2}{2}}\left\|{}^*[A^{-\frac{1}{2}}]{}^*[B^{\frac{1}{2}}]\right\|_{\mathbb{S}^N}\\
&\quad\times |B^{\frac{1}{2}}\nabla y_B||A^{\frac{1}{2}}\nabla y_A - A^{\frac{1}{2}}\nabla y_B|\,dx\\
&\leqslant \left\|{}^*[B^{-\frac{1}{2}}](A-B)B^{-\frac{1}{2}}\right\|^{\frac{p-2}{2}}_{L^\infty(\Omega;\mathbb{S}^N)}\\
&\quad\times \left\|B^{\frac{1}{2}}A^{-\frac{1}{2}}\right\|_{L^\infty(\Omega;\mathbb{S}^N)}\int_\Omega |B^{\frac{1}{2}}\nabla y_B|^{p-1}|A^{\frac{1}{2}}\nabla y_A - A^{\frac{1}{2}}\nabla y_B|\,dx\\
&\overset{\text{by (3.10)}}{\leqslant} \left\|{}^*[B^{-\frac{1}{2}}](A-B)B^{-\frac{1}{2}}\right\|^{\frac{p-2}{2}}_{L^\infty(\Omega;\mathbb{S}^N)}\left\|B^{\frac{1}{2}}A^{-\frac{1}{2}}\right\|_{L^\infty(\Omega;\mathbb{S}^N)}\\
&\quad\times \|y_B\|^{p-1}_{H^p_B(\Omega)}\|y_A-y_B\|_{H^p_A(\Omega)}, \qquad (3.18)
\end{aligned}$$

provided $B^{\frac{1}{2}}A^{-\frac{1}{2}} \in L^\infty(\Omega;\mathbb{S}^N)$ and ${}^*[B^{-\frac{1}{2}}](A-B)B^{-\frac{1}{2}} \in L^\infty(\Omega;\mathbb{S}^N)$.

If $p > 4$, then

$$
\begin{aligned}
|I_3| &\leqslant \frac{p-2}{2} \int_\Omega \left(|(A\nabla y_B, \nabla y_B)_{\mathbb{R}^N}| + |(B\nabla y_B, \nabla y_B)_{\mathbb{R}^N}|\right)^{\frac{p-4}{2}} \\
&\quad \times |\left((A-B)\nabla y_B, \nabla y_B\right)_{\mathbb{R}^N}| \cdot |{}^*[A^{-\frac12}]B^{\frac12}\nabla y_B| \\
&\quad \times |A^{\frac12}\nabla y_A - A^{\frac12}\nabla y_B|\, dx \\
&\leqslant \frac{p-2}{2}\left(1 + \left\| {}^*[B^{-\frac12}]AB^{-\frac12}\right\|_{L^\infty(\Omega;\mathbb{S}^N)}\right)^{\frac{p-4}{2}} \left\| B^{\frac12}A^{-\frac12}\right\|_{L^\infty(\Omega;\mathbb{S}^N)} \\
&\quad \times \int_\Omega |B^{\frac12}\nabla y_B|^{p-4} |\left((A-B)\nabla y_B, \nabla y_B\right)_{\mathbb{R}^N}| \\
&\qquad \times |B^{\frac12}\nabla y_B| \cdot |A^{\frac12}\nabla y_A - A^{\frac12}\nabla y_B|\, dx \\
&\leqslant \frac{p-2}{2}\left(1 + \left\| {}^*[B^{-\frac12}]AB^{-\frac12}\right\|_{L^\infty(\Omega;\mathbb{S}^N)}\right)^{\frac{p-4}{2}} \left\| B^{\frac12}A^{-\frac12}\right\|_{L^\infty(\Omega;\mathbb{S}^N)} \\
&\quad \times \left\| {}^*[B^{-\frac12}](A-B)B^{-\frac12}\right\|_{L^\infty(\Omega;\mathbb{S}^N)} \\
&\quad \times \int_\Omega |B^{\frac12}\nabla y_B|^{p-1} |A^{\frac12}\nabla y_A - A^{\frac12}\nabla y_B|\, dx \\
&\leqslant \mathrm{const} \left\| {}^*[B^{-\frac12}](A-B)B^{-\frac12}\right\|_{L^\infty(\Omega;\mathbb{S}^N)} \|y_B\|^{p-1}_{H^p_B(\Omega)} \|y_A - y_B\|_{H^p_A(\Omega)}. \qquad (3.19)
\end{aligned}
$$

Gathering (3.8), (3.9) and estimates derived before, we obtain for $p \in [2, 4]$

$$
\begin{aligned}
C_p \|y_A - y_B\|^{p-1}_{H^p_A(\Omega)} &\leqslant \left\| {}^*[B^{-\frac12}]AB^{-\frac12}\right\|^{\frac{p-2}{2}}_{L^\infty(\Omega;\mathbb{S}^N)} \left\| B^{\frac12}A^{-\frac12}\right\|_{L^\infty(\Omega;\mathbb{S}^N)} \\
&\quad \times \|g\|_{L^\infty(\Omega;\mathbb{R}^N)} \|\alpha^{-1}\|_{L^q(\Omega)} \left\| {}^*[B^{-\frac12}](A-B)B^{-\frac12}\right\|_{L^\infty(\Omega;\mathbb{S}^N)} \\
&\quad + \left\| B^{\frac12}A^{-\frac12}\right\|_{L^\infty(\Omega;\mathbb{S}^N)} \|y_B\|^{p-1}_{H^p_B(\Omega)} \left\| {}^*[B^{-\frac12}](A-B)B^{-\frac12}\right\|^{\frac{p-2}{2}}_{L^\infty(\Omega;\mathbb{S}^N)} \\
&\leqslant \left\| B^{\frac12}A^{-\frac12}\right\|_{L^\infty(\Omega;\mathbb{S}^N)} \|g\|_{L^\infty(\Omega;\mathbb{R}^N)} \|\alpha^{-1}\|_{L^q(\Omega)} \\
&\quad \times \left\| {}^*[B^{-\frac12}](A-B)B^{-\frac12}\right\|^{\frac{p-2}{2}}_{L^\infty(\Omega;\mathbb{S}^N)} \\
&\quad \times \left[\left\| {}^*[B^{-\frac12}]AB^{-\frac12}\right\|^{\frac{p-2}{2}}_{L^\infty(\Omega;\mathbb{S}^N)} \left\| {}^*[B^{-\frac12}](A-B)B^{-\frac12}\right\|^{\frac{4-p}{2}}_{L^\infty(\Omega;\mathbb{S}^N)} + 1 \right], \qquad (3.20)
\end{aligned}
$$

and for $p > 4$

$$
\begin{aligned}
&C_p \|y_A - y_B\|^{p-1}_{H^p_A(\Omega)} \\
&\quad \leqslant \left\| {}^*[A^{-\frac12}](A-B)B^{-\frac12}\right\|_{L^\infty(\Omega;\mathbb{S}^N)} \left\| {}^*[B^{-\frac12}]AB^{-\frac12}\right\|^{\frac{p-2}{2}}_{L^\infty(\Omega;\mathbb{S}^N)} \\
&\quad \times \|g\|_{L^\infty(\Omega;\mathbb{R}^N)} \|\alpha^{-1}\|_{L^q(\Omega)} + \left\| {}^*[B^{-\frac12}](A-B)B^{-\frac12}\right\|_{L^\infty(\Omega;\mathbb{S}^N)}
\end{aligned}
$$

$$
\begin{aligned}
&\times \|y_B\|^{p-1}_{H^p_B(\Omega)} \frac{p-2}{2}\left(1+\left\| {}^*[B^{-\frac12}]AB^{-\frac12}\right\|_{L^\infty(\Omega;\mathbb{S}^N)}\right)^{\frac{p-4}{2}} \left\|B^{\frac12}A^{-\frac12}\right\|_{L^\infty(\Omega;\mathbb{S}^N)}\\
&\overset{\text{by (3.6), (3.12), (3.16)}}{\leqslant} \|g\|_{L^\infty(\Omega;\mathbb{R}^N)}\|\alpha^{-1}\|_{L^q(\Omega)}\left[\left\|{}^*[B^{-\frac12}](A-B)B^{-\frac12}\right\|_{L^\infty(\Omega;\mathbb{S}^N)}\right.\\
&\times \left\| {}^*[B^{\frac12}A^{-\frac12}]\right\|_{L^\infty(\Omega;\mathbb{S}^N)} \left\|{}^*[B^{-\frac12}]AB^{-\frac12}\right\|^{\frac{p-2}{2}}_{L^\infty(\Omega;\mathbb{S}^N)} + \left\|B^{\frac12}A^{-\frac12}\right\|_{L^\infty(\Omega;\mathbb{S}^N)}\\
&\times \frac{p-2}{2}\left(1+\left\| {}^*[B^{-\frac12}]AB^{-\frac12}\right\|_{L^\infty(\Omega;\mathbb{S}^N)}\right)^{\frac{p-4}{2}}\\
&\times \left.\left\|{}^*[B^{-\frac12}](A-B)B^{-\frac12}\right\|_{L^\infty(\Omega;\mathbb{S}^N)}\right]\\
&\leqslant \frac{p}{2}\|g\|_{L^\infty(\Omega;\mathbb{R}^N)}\|\alpha^{-1}\|_{L^q(\Omega)}\left\|B^{\frac12}A^{-\frac12}\right\|_{L^\infty(\Omega;\mathbb{S}^N)}\times\\
&\times\left(1+\left\| {}^*[B^{-\frac12}]AB^{-\frac12}\right\|_{L^\infty(\Omega;\mathbb{S}^N)}\right)^{\frac{p-2}{2}} \left\|{}^*[B^{-\frac12}](A-B)B^{-\frac12}\right\|_{L^\infty(\Omega;\mathbb{S}^N)} \qquad (3.21)
\end{aligned}
$$

□

To proceed further, we need the following result.

Lemma 3.2. *For any $\varphi \in H^p_A(\Omega)$ such that $\|\varphi\|_{H^p_A(\Omega)} = 1$ and arbitrary fixed element $y \in H^p_B(\Omega)$, we have*

$$
\begin{aligned}
&|\langle \mathcal{A}_A(y_B) - \mathcal{A}_A(y_A), \varphi\rangle_{-1;1}|\\
&\leqslant \begin{cases}
\left\| {}^*[B^{-\frac12}]AB^{-\frac12}\right\|^{\frac{p-2}{2}}_{L^\infty(\Omega;\mathbb{S}^N)} \|y\|^{p-2}_{H^p_B(\Omega)}\|y_B-y_A\|_{H^p_A(\Omega)} & \\
\quad +\|y_B-y_A\|^{\frac{p-2}{2}}_{H^p_A(\Omega)}\|y_B+y_A\|^{\frac{p-2}{2}}_{H^p_A(\Omega)}\|y_A\|_{H^p_A(\Omega)}, & \forall\, p\in[2,4];\\
\left\| {}^*[B^{-\frac12}]AB^{-\frac12}\right\|^{\frac{p-2}{2}}_{L^\infty(\Omega;\mathbb{S}^N)} \|y\|^{p-2}_{H^p_B(\Omega)}\|y_B-y_A\|_{H^p_A(\Omega)} & \\
\quad +\dfrac{p-2}{2}\left(\|y_B\|_{H^p_A(\Omega)}+\|y_A\|_{H^p_A(\Omega)}\right)^{p-2}\|y_B-y_A\|_{H^p_A(\Omega)}, & \forall\, p>4,
\end{cases}
\end{aligned}
\qquad (3.22)
$$

where $y_A = \mathcal{R}_A(f)$, $y_B = \mathcal{R}_B(f)$, $f = \mathcal{A}_B(y)$, $\|y_B - y_A\|_{H^p_A(\Omega)} < +\infty$ and

$$
\|y_B\|_{H^p_A(\Omega)} \leqslant \left\|{}^*[B^{-\frac12}]AB^{-\frac12}\right\|^{\frac12}_{L^\infty(\Omega;\mathbb{S}^N)}\|y_B\|_{H^p_B(\Omega)}.
$$

Proof. Let us fix two matrices $A, B \in \mathcal{M}(\Omega)$ such that ${}^t[B^{-\frac12}]AB^{-\frac12} \in L^\infty(\Omega;\mathbb{S}^N)$. Then (see (3.13)) $y \in H^p_A(\Omega)$ and, therefore, the following expression makes sense

$$
\begin{aligned}
&\langle \mathcal{A}_A(y) - \mathcal{A}_B(y), v\rangle_{-1;1}\\
&\quad = \int_\Omega \left(|A^{\frac12}\nabla y|^{p-2}A\nabla y - |B^{\frac12}\nabla y|^{p-2}B\nabla y, \nabla v\right)_{\mathbb{R}^N} dx,\ \forall\, v\in C^\infty_0(\Omega). \qquad (3.23)
\end{aligned}
$$

On the other hand, provided $\mathcal{A}_B(y) \in \left(H^p_A(\Omega)\right)^*$, we see that

$$\mathcal{A}_A(y) - \mathcal{A}_B(y) = \mathcal{A}_A\big(\underbrace{\mathcal{R}_B(\mathcal{A}_B(y))}_{y_B}\big) - \mathcal{A}_A\big(\underbrace{\mathcal{R}_A(\mathcal{A}_B(y))}_{y_A}\big) \tag{3.24}$$

To estimate the norm of the difference $\mathcal{A}_A(y_B) - \mathcal{A}_A(y_A)$ we make the following transformations

$$\begin{aligned}
&\langle \mathcal{A}_A(y_B) - \mathcal{A}_A(y_A), \varphi\rangle_{-1;1} \\
&= \int_\Omega \Big(|(A\nabla y_B, \nabla y_B)_{\mathbb{R}^N}|^{\frac{p-2}{2}} A\nabla y_B - |(A\nabla y_A, \nabla y_A)_{\mathbb{R}^N}|^{\frac{p-2}{2}} A\nabla y_A, \nabla\varphi\Big)_{\mathbb{R}^N} dx \\
&= \int_\Omega |(A\nabla y_B, \nabla y_B)_{\mathbb{R}^N}|^{\frac{p-2}{2}} \left(A(\nabla y_B - \nabla y_A), \nabla\varphi\right)_{\mathbb{R}^N} dx \\
&+ \int_\Omega \Big((A\nabla y_B, \nabla y_B)_{\mathbb{R}^N}|^{\frac{p-2}{2}} - (A\nabla y_A, \nabla y_A)_{\mathbb{R}^N}|^{\frac{p-2}{2}}\Big) (A\nabla y_A, \nabla\varphi)_{\mathbb{R}^N} dx \\
&= I_1 + I_2.
\end{aligned} \tag{3.25}$$

Then, by Hölder inequality, noting that $\|\varphi\|_{H^p_A(\Omega)} = 1$, we deduce

$$\begin{aligned}
|I_1| &\leqslant \int_\Omega \left\| {}^*[B^{-\frac12}]AB^{-\frac12}\right\|_{\mathbb{S}^N}^{\frac{p-2}{2}} |B^{\frac12}\nabla y_B|^{p-2} |A^{\frac12}(\nabla y_B - \nabla y_A)|\, |A^{\frac12}\nabla\varphi| dx \\
&\leqslant \left\| {}^*[B^{-\frac12}]AB^{-\frac12}\right\|_{L^\infty(\Omega;\mathbb{S}^N)}^{\frac{p-2}{2}} \|y_B\|^{p-2}_{H^p_B(\Omega)} \|y_B - y_A\|_{H^p_A(\Omega)} \|\varphi\|_{H^p_A(\Omega)} \\
&\overset{\text{by (3.24)}}{=} \left\| {}^*[B^{-\frac12}]AB^{-\frac12}\right\|_{L^\infty(\Omega;\mathbb{S}^N)}^{\frac{p-2}{2}} \|y\|^{p-2}_{H^p_B(\Omega)} \|y_B - y_A\|_{H^p_A(\Omega)}
\end{aligned}$$

and

$$|I_2| \leqslant \int_\Omega \Big| |(A\nabla y_B, \nabla y_B)_{\mathbb{R}^N}|^{\frac{p-2}{2}} - |(A\nabla y_A, \nabla y_A)_{\mathbb{R}^N}|^{\frac{p-2}{2}}\Big|\, |A^{\frac12}\nabla y_A|\, |A^{\frac12}\nabla\varphi| dx.$$

Hence, for $p \in [2,4]$, we have (see $(3.17)_2$)

$$\begin{aligned}
|I_2| &\leqslant \int_\Omega |(A(\nabla y_B - \nabla y_A), \nabla y_B + \nabla y_A)_{\mathbb{R}^N}|^{\frac{p-2}{2}} |A^{\frac12}\nabla y_A|\, |A^{\frac12}\nabla\varphi| dx \\
&\leqslant \int_\Omega \Big|A^{\frac12}(\nabla y_B - \nabla y_A)\Big|^{\frac{p-2}{2}} \Big|A^{\frac12}(\nabla y_B + \nabla y_A)\Big|^{\frac{p-2}{2}} |A^{\frac12}\nabla y_A|\, |A^{\frac12}\nabla\varphi| dx \\
&\leqslant \left\{ \text{with } p_1 = p_2 = \frac{2p}{p-2},\ p_3 = p_4 = p,\ \sum_{i=1}^4 p_i = 1 \right\} \\
&\leqslant \|y_B - y_A\|^{\frac{p-2}{2}}_{H^p_A(\Omega)} \|y_B + y_A\|^{\frac{p-2}{2}}_{H^p_A(\Omega)} \|y_A\|_{H^p_A(\Omega)} \|\varphi\|_{H^p_A(\Omega)},
\end{aligned}$$

where $\|\varphi\|_{H^p_A(\Omega)} = 1$ and

$$\begin{aligned}
\|y_B - y_A\|^{\frac{p-2}{2}}_{H^p_A(\Omega)} &= \left(\int_\Omega |(A\nabla y_B + A\nabla y_A, \nabla y_B + \nabla y_A)_{\mathbb{R}^N}|^{\frac{p}{2}}\,dx\right)^{\frac{p-2}{2p}} \\
&\leqslant \left(\int_\Omega 2^{p-1}\left(|A^{\frac{1}{2}}\nabla y_B|^p + |A^{\frac{1}{2}}\nabla y_A|^p\right)dx\right)^{\frac{p-2}{2p}} \\
&\leqslant \left(\int_\Omega 2^{p-1}\left(\left\|{}^*[B^{-\frac{1}{2}}]AB^{-\frac{1}{2}}\right\|^{\frac{p}{2}}_{\mathbb{S}^N} |B^{\frac{1}{2}}\nabla y_B|^p + |A^{\frac{1}{2}}\nabla y_A|^p\right)dx\right)^{\frac{p-2}{2p}} \\
&\leqslant \left(2^{p-1}\cdot\max\left\{\left\|{}^*[B^{-\frac{1}{2}}]AB^{-\frac{1}{2}}\right\|^{\frac{1}{2}}_{L^\infty(\Omega;\mathbb{S}^N)}, 1\right\}\right)^{\frac{p-2}{2}} \\
&\quad\times\left(\|y_B\|_{H^p_B(\Omega)} + \|y_A\|_{H^p_A(\Omega)}\right)^{\frac{p-2}{2}} < +\infty. \qquad (3.26)
\end{aligned}$$

For the case $p > 4$, we have (see $(3.17)_1$)

$$\begin{aligned}
|I_2| &\leqslant \frac{p-2}{2}\int_\Omega \left(|(A\nabla y_B, \nabla y_B)_{\mathbb{R}^N}| + |(A\nabla y_A, \nabla y_A)_{\mathbb{R}^N}|\right)^{\frac{p-4}{2}} \\
&\quad\times |A^{\frac{1}{2}}(\nabla y_B - \nabla y_A)|\,|A^{\frac{1}{2}}(\nabla y_B + \nabla y_A)||A^{\frac{1}{2}}\nabla y_A|\,|A^{\frac{1}{2}}\nabla\varphi|dx \\
&\leqslant \frac{p-2}{2}\int_\Omega \left(|A^{\frac{1}{2}}\nabla y_B| + |A^{\frac{1}{2}}\nabla y_A|\right)^{p-3} \\
&\quad\times\left|A^{\frac{1}{2}}(\nabla y_B - \nabla y_A)\right|\,|A^{\frac{1}{2}}\nabla y_A|\,|A^{\frac{1}{2}}\nabla\varphi|\,dx \\
&\leqslant \frac{p-2}{2}\int_\Omega \left(|A^{\frac{1}{2}}\nabla y_B| + |A^{\frac{1}{2}}\nabla y_A|\right)^{p-2}\left|A^{\frac{1}{2}}(\nabla y_B - \nabla y_A)\right|\,|A^{\frac{1}{2}}\nabla\varphi|dx \\
&\leqslant \left\{p_1 = \frac{p}{p-2},\ p_2, p_3 = p\right\} \\
&\leqslant \left(\int_\Omega \left(|A^{\frac{1}{2}}\nabla y_B| + |A^{\frac{1}{2}}\nabla y_A|\right)^p dx\right)^{\frac{p-2}{p}} \|y_B - y_A\|_{H^p_A(\Omega)}\|\varphi\|_{H^p_A(\Omega)} \\
&\leqslant \frac{p-2}{2}\left(\|y_B\|_{H^p_A(\Omega)} + \|y_A\|_{H^p_A(\Omega)}\right)^{p-2}\|y_B - y_A\|_{H^p_A(\Omega)}\|\varphi\|_{H^p_A(\Omega)}.
\end{aligned}$$

It remains to take into account that

$$\begin{aligned}
\|y_B\|_{H^p_A(\Omega)} &= \left(\int_\Omega |(A\nabla y_B, \nabla y_B)_{\mathbb{R}^N}|^{\frac{p}{2}} dx\right)^{\frac{1}{p}} \\
&\leqslant \left\|{}^*[B^{-\frac{1}{2}}]AB^{-\frac{1}{2}}\right\|^{\frac{1}{2}}_{L^\infty(\Omega;\mathbb{S}^N)} \|y_B\|_{H^p_B(\Omega)}. \qquad (3.27)
\end{aligned}$$

Hence,

$$\begin{aligned}
|I_2| &\leqslant \frac{p-2}{2}\left(\max\left\{1, \left\|{}^*[B^{-\frac{1}{2}}]AB^{-\frac{1}{2}}\right\|^{\frac{1}{2}}_{L^\infty(\Omega;\mathbb{S}^N)}\right\}\right)^{p-2} \\
&\quad\times\left(\|y_B\|_{H^p_A(\Omega)} + \|y_A\|_{H^p_A(\Omega)}\right)^{p-2}\|y_B - y_A\|_{H^p_A(\Omega)}.
\end{aligned}$$

In view of the Lemma preconditions, we have $f = \mathcal{A}_B(y) \in \left(H^p_B(\Omega)\right)^*$. Since

$$\|y\|_{H^p_B(\Omega)} \leqslant \left(\int_\Omega |(B\nabla y, \nabla y)^{\frac{p}{2}}_{\mathbb{R}^N}|\right)^{\frac{1}{p}} \overset{\text{by (1.4)}}{\leqslant} \left(\int_\Omega \left(\beta|\nabla y|^2\right)^{\frac{p}{2}} dx\right)^{\frac{1}{p}}$$
$$\leqslant \|\beta\|^{\frac{1}{2}}_{L^\infty(\Omega)} \|y\|_{W^{1,p}_0(\Omega)},$$

it follows that $W^{1,p}_0(\Omega) \subset H^p_B(\Omega)$ with continuous embedding. Hence, $\left(H^p_B(\Omega)\right)^*$ is continuously embedded into $W^{-1,q}(\Omega)$ and in view of the estimate

$$\langle f, y\rangle_{W^{-1,q}(\Omega);W^{1,p}_0(\Omega)} = \langle f, y\rangle_{\left(H^p_B(\Omega)\right)^*;H^p_B(\Omega)} \leqslant \|f\|_{\left(H^p_B(\Omega)\right)^*} \|y\|_{H^p_B(\Omega)}$$
$$\leqslant \|\beta\|^{\frac{1}{2}}_{L^\infty(\Omega)} \|f\|_{\left(H^p_B(\Omega)\right)^*} \|y\|_{W^{1,p}_0(\Omega)} \ \forall\, y \in W^{1,p}_0(\Omega),$$

we can conclude

$$\|f\|_{W^{-1,q}(\Omega)} \leqslant \|\beta\|^{\frac{1}{2}}_{L^\infty(\Omega)} \|f\|_{\left(H^p_B(\Omega)\right)^*}.$$

Thus, (see (3.24)), for a given element $y \in H^p_A(\Omega)$ we have $y \in H^p_B(\Omega)$ provided ${}^*[B^{-\frac{1}{2}}]AB^{-\frac{1}{2}} \in L^\infty(\Omega; \mathbb{S}^N)$ and, therefore,

$$f = \mathcal{A}_B(y) \in \left(H^p_A(\Omega)\right)^* \cap \left(H^p_B(\Omega)\right)^*. \tag{3.28}$$

Then

$$\|y_B\|_{H^p_B(\Omega)} = \|\mathcal{R}_B(\underbrace{\mathcal{A}_B(y)}_{f})\|_{H^p_B(\Omega)} \leqslant \|f\|^{\frac{1}{p-1}}_{\left(H^p_B(\Omega)\right)^*} < +\infty$$

and

$$\|y_A\|_{H^p_A(\Omega)} = \|\mathcal{R}_A(\mathcal{A}_B(y))\|_{H^p_A(\Omega)} \leqslant \|f\|^{\frac{1}{p-1}}_{\left(H^p_A(\Omega)\right)^*} < +\infty.$$

As follows from (3.25) and estimates derived above, we finally can give the desired conclusion. $\square$

4. Main Result.

Our next intention is to evaluate the expression (3.23) provided $v = y$, where $y \in H^p_B(\Omega)$ is an arbitrary element. Since ${}^*[B^{-\frac{1}{2}}]AB^{-\frac{1}{2}} \in L^\infty(\Omega; \mathbb{S}^N)$ it follows from (3.13) that $y \in H^p_A(\Omega)$. Hence (3.23) leads to the following transformations

$$\langle \mathcal{A}_A(y) - \mathcal{A}_B(y), y\rangle_{-1;1} = \int_\Omega \left(|(B\nabla y, \nabla y)_{\mathbb{R}^N}|^{\frac{p}{2}} - |(B\nabla y, \nabla y)_{\mathbb{R}^N}|^{\frac{p}{2}}\right) dx$$
$$= \int_\Omega \left(|B^{\frac{1}{2}}\nabla y|^p - |({}^*[B^{-\frac{1}{2}}]AB^{-\frac{1}{2}} B^{\frac{1}{2}}\nabla y, B^{\frac{1}{2}}\nabla y)_{\mathbb{R}^N}|^{\frac{p}{2}}\right) dx$$
$$\geqslant \int_\Omega \underbrace{\left(1 - \left\| {}^*[B^{-\frac{1}{2}}]AB^{-\frac{1}{2}} \right\|^{\frac{p}{2}}_{\mathbb{S}^N}\right)}_{\gamma(x)} |B^{\frac{1}{2}}\nabla y|^p dx = I. \tag{4.1}$$

Let us define the transformation $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ such that

$$B^{\frac{1}{2}} = (\det \mathcal{D}\Phi)^{\frac{1}{p}} \; {}^*[\mathcal{D}\Phi]^{-1},$$

where $\mathcal{D}\Phi$ is the Jacobian matrix of Φ, and make the following change of variables

$$y \to \widetilde{y} \circ \Phi^{-1}.$$

Then, from (4.1) we deduce

$$I = \int_{\Phi^{-1}(\Omega)} \gamma(\Phi(x)) |\nabla \widetilde{y}|^p dx.$$

Remark 4.1. Since $|B^{\frac{1}{2}} \nabla y|^2 = (B\nabla y, \nabla y)_{\mathbb{R}^N} \geqslant \alpha^2(x)|\nabla y|^2$, where $\alpha^{-q} \in L^1(\Omega)$, it follows that $\det \mathcal{D}\Phi$ can reach zero value on a set with zero Lebesgue measure. Hence, we can admit the existence of a set $\Lambda \subset \Omega$ with $|\Lambda| = 0$ such that the mapping $\Phi : \mathbb{R}^N \setminus \Lambda \to \mathbb{R}^N$ is a gomeomorphizm. In other words, for almost all $x^0 \in \Omega$ there exists a point $\widetilde{x}^0 \in \Phi^{-1}(\Omega)$ such that $x^0 = \Phi(\widetilde{x}^0)$.

Lemma 4.1. *Let* $\gamma = 1 - \left\| {}^*[B^{-\frac{1}{2}}] A B^{-\frac{1}{2}} \right\|^{\frac{p}{2}}_{\mathbb{S}^N} \in L^\infty(\Omega)$*. Then for almost all* $x_0 \in \Omega$ *there exists a sequence* $\{y_{\varepsilon,x_0}\}_{\varepsilon>0} \subset H^p_B(\Omega)$ *such that*

$$\|y_{\varepsilon,x_0}\|_{H^p_B(\Omega)} = 1, \; \forall\, \varepsilon > 0$$

and

$$\lim_{\varepsilon\to 0} \int_\Omega \gamma(x) |B^{\frac{1}{2}} \nabla y_{\varepsilon,x_0}|^p dx = \gamma(x_0).$$

Proof. We construct this sequence as follows

$$y_{\varepsilon,x_0} = \widetilde{y}_{\varepsilon,x_0} \circ \Phi^{-1}(x),$$

where

$$\widetilde{y}_{\varepsilon,x_0}(x) = \frac{1}{\sqrt[p]{|B_\varepsilon(\widetilde{x}_0)|}} \varphi_\varepsilon(|x - \widetilde{x}_0|),$$

$$\widetilde{x}_0 = \Phi^{-1}(x_0), \; \varphi_\varepsilon(r) = \begin{cases} 0, & r < 0, \\ r, & r \in [0, \varepsilon], \\ \varepsilon, & r > \varepsilon, \end{cases}$$

$$B_\varepsilon(\widetilde{x}_0) = \left\{ x \in \mathbb{R}^N : \, |x - \widetilde{x}_0| < \varepsilon \right\}.$$

Then

$$\int_\Omega \gamma(x) |B^{\frac{1}{2}} \nabla y_{\varepsilon,x_0}|^p dx = \int_{\Phi^{-1}(\Omega)} \gamma(\Phi(x)) |\nabla \widetilde{y}_{\varepsilon,x_0}|^p \, dx =$$

$$\left\{\begin{array}{c} \nabla \widetilde{y}_{\varepsilon,x_0} = \chi_{B_\varepsilon(\widetilde{x}_0)(x)} \dfrac{1}{\sqrt[p]{|B_\varepsilon(\widetilde{x}_0)|}} \dfrac{x-\widetilde{x}_0}{|x-\widetilde{x}_0|},\ |\nabla \widetilde{y}_{\varepsilon,x_0}|^p = \dfrac{1}{|B_\varepsilon(\widetilde{x}_0)|}, \\ \|\widetilde{y}_{\varepsilon,x_0}\|_{W_0^{1,p}(\Phi^{-1}(\Omega))} \equiv \|y_{\varepsilon,x_0}\|^p_{H_B^p(\Omega)} = \int\limits_{\Phi^{-1}(\Omega)} \dfrac{1}{|B_\varepsilon(\widetilde{x}_0)|} \chi_{B_\varepsilon(\widetilde{x}_0)(x)} dx \\ = \dfrac{1}{|B_\varepsilon(\widetilde{x}_0)|} \displaystyle\int_{B_\varepsilon(\widetilde{x}_0)} dx = 1,\ \forall\, \varepsilon > 0 \end{array}\right\}$$

$$= \frac{1}{|B_\varepsilon(\widetilde{x}_0)|} \int_{B_\varepsilon(\widetilde{x}_0)} \gamma(\Phi(x)) dx.$$

Hence, by Lebesgue Differentiation Theorem, we obtain

$$\lim_{\varepsilon\to 0} \int_\Omega \gamma(x) |B^{\frac{1}{2}} \nabla y_{\varepsilon,x_0}|^p dx = \lim_{\varepsilon\to 0} \frac{1}{|B_\varepsilon(\widetilde{x}_0)|} \int_{B_\varepsilon(\widetilde{x}_0)} \gamma(\Phi(x)) dx$$
$$= \gamma(\Phi(\widetilde{x}_0)) = \gamma(x_0).$$

□

Combining the result of this lemma with (4.1), we arrive at the following obvioius conclusion:

Theorem 4.1. *Let A and B be given elements from $\mathcal{M}(\Omega)$ such that*

$$^*[B^{-\frac{1}{2}}] A B^{-\frac{1}{2}} \in L^\infty(\Omega; \mathbb{S}^N) \quad \text{and} \quad \left\| {}^*[B^{-\frac{1}{2}}] A B^{-\frac{1}{2}} \right\|^{\frac{p}{2}}_{L^\infty(\Omega;\mathbb{S}^N)} \leqslant 1. \tag{4.2}$$

Assume also that for some $C > 0$ $\langle \mathcal{A}_A(y) - \mathcal{A}_B(y), y\rangle_{-1;1} \leqslant C$ for all $y \in H^p_B(\Omega)$ such that $\|y\|_{H^p_B(\Omega)} = 1$. Then

$$1 - \left\| {}^*[B^{-\frac{1}{2}}] A B^{-\frac{1}{2}} \right\|^{\frac{p}{2}}_{L^\infty(\Omega;\mathbb{S}^N)} \leqslant C. \tag{4.3}$$

Proof. As follows from Lemma 4.1, for almost all $x_0 \in \Omega$ there exists a sequence $\{y_{\varepsilon,x_0}\}_{\varepsilon>0} \subset H^p_B(\Omega)$ such that $\|y_{\varepsilon,x_0}\|^p_{H^p_B(\Omega)} = 1$, $\forall\, \varepsilon > 0$ and

$$\begin{aligned} C &\geqslant \liminf_{\varepsilon\to 0} \langle \mathcal{A}_A(y) - \mathcal{A}_B(y), y\rangle_{-1;1} \\ &\geqslant \lim_{\varepsilon\to 0} \int_\Omega \left(1 - \left\| {}^*[B^{-\frac{1}{2}}] A B^{-\frac{1}{2}} \right\|^{\frac{p}{2}}_{\mathbb{S}^N}\right) |B^{\frac{1}{2}} \nabla y_{\varepsilon,x_0}|^p dx \\ &= 1 - \left\| {}^*[B^{-\frac{1}{2}}(x_0)] A(x_0) B^{-\frac{1}{2}(x_0)} \right\|^{\frac{p}{2}}_{\mathbb{S}^N}. \end{aligned}$$

Therefore, in view of (4.2), the estimate

$$0 < 1 - \left\| {}^*[B^{-\frac{1}{2}}(x_0)] A(x_0) B^{-\frac{1}{2}(x_0)} \right\|^{\frac{p}{2}}_{\mathbb{S}^N} \leqslant C \text{ for a.a. } x_0 \in \Omega$$

implies (4.3). □

As a result, Theorem 4.1 lead us to the conclusion stated in Theorem 1.1, i.e.

1. for $p = 2$

$$1 - \left\| {}^*[B^{-\frac{1}{2}}]AB^{-\frac{1}{2}} \right\|^{\frac{p}{2}}_{L^\infty(\Omega;\mathbb{S}^N)} \leqslant \|\mathcal{R}_B(f) - \mathcal{R}_A(f)\|_{H^2_A(\Omega)};$$

2. for $p \in (2;4]$

$$\begin{aligned} 1- \left\| {}^*[B^{-\frac{1}{2}}]AB^{-\frac{1}{2}} \right\|^{\frac{p}{2}}_{L^\infty(\Omega;\mathbb{S}^N)} &\overset{\text{by (3.22),(4.2)}}{\leqslant} \|\mathcal{R}_B(f) - \mathcal{R}_A(f)\|_{H^p_A(\Omega)} \\ &+ \|\mathcal{R}_B(f) + \mathcal{R}_A(f)\|^{\frac{p-2}{2}}_{H^p_A(\Omega)} \|\mathcal{R}_A(f)\|_{H^p_A(\Omega)} \|\mathcal{R}_B(f) - \mathcal{R}_A(f)\|_{H^p_A(\Omega)} \\ &= \left[1 + \|\mathcal{R}_B(f) + \mathcal{R}_A(f)\|^{\frac{p-2}{2}}_{H^p_A(\Omega)} \|\mathcal{R}_A(f)\|_{H^p_A(\Omega)} \right] \\ &\times \|\mathcal{R}_B(f) - \mathcal{R}_A(f)\|_{H^p_A(\Omega)}; \end{aligned}$$

3. for $p > 4$

$$\begin{aligned} 1 - &\left\| {}^*[B^{-\frac{1}{2}}]AB^{-\frac{1}{2}} \right\|^{\frac{p}{2}}_{L^\infty(\Omega;\mathbb{S}^N)} \\ &\leqslant \left(1 + \frac{p-2}{2} \left(\|y_A\|_{H^p_A(\Omega)} + 1 \right)^{p-2} \right) \|\mathcal{R}_B(f) - \mathcal{R}_A(f)\|_{H^p_A(\Omega)} \\ &= \left[1 + \frac{p-2}{2} \left(\|\mathcal{R}_A(f)\|_{H^p_A(\Omega)} + 1 \right)^{p-2} \right] \|\mathcal{R}_B(f) - \mathcal{R}_A(f)\|_{H^p_A(\Omega)}, \end{aligned}$$

Remark 4.2. At the end of the paper, we indicate the following estimate

$$\begin{aligned} 1- \left\| {}^*[B^{-\frac{1}{2}}]AB^{-\frac{1}{2}} \right\|^{\frac{p}{2}}_{L^\infty(\Omega;\mathbb{S}^N)} &\geqslant 1 - \|B^{-\frac{1}{2}}\|^p_{L^\infty(\Omega;\mathbb{S}^N)} \|A\|^{\frac{p}{2}}_{L^\infty(\Omega;\mathbb{S}^N)} \\ &= \|B^{-\frac{1}{2}}\|^p_{L^\infty(\Omega;\mathbb{S}^N)} \left[\|B^{-\frac{1}{2}}\|^{-p}_{L^\infty(\Omega;\mathbb{S}^N)} - \|A\|^{\frac{p}{2}}_{L^\infty(\Omega;\mathbb{S}^N)} \right], \end{aligned}$$

where

$$\|B^{-\frac{1}{2}}\|^{-p}_{L^\infty(\Omega;\mathbb{S}^N)} \leqslant \|B\|^{\frac{p}{2}}_{L^\infty(\Omega;\mathbb{S}^N)}.$$

Indeed, the last inequality is a direct consequence of the following chain of estimates

$$\begin{aligned} \left(B^{-\frac{1}{2}} B^{\frac{1}{2}} \xi, B^{-\frac{1}{2}} B^{\frac{1}{2}} \xi \right)_{\mathbb{R}^N} &= \left({}^*[B^{-\frac{1}{2}}] B^{-\frac{1}{2}} B^{\frac{1}{2}} \xi, B^{\frac{1}{2}} \xi \right)_{\mathbb{R}^N} \\ &\leqslant \|B^{-\frac{1}{2}}(x)\|^2_{\mathbb{S}^N} |B^{\frac{1}{2}} \xi|^2 = \|B^{-\frac{1}{2}}(x)\|^2_{\mathbb{S}^N} \left(B\xi, \xi \right)_{\mathbb{R}^N} \\ &\leqslant \|B^{-\frac{1}{2}}(x)\|^2_{\mathbb{S}^N} \|B(x)\|_{\mathbb{S}^N} |\xi|^2. \end{aligned}$$

References

1. R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.
2. P. Binev, A. Cohen, W. Dahmen, R. DeVore, G. Petrova, P. Wojtaszczyk, *Convergence rates for greedy algorithms in reduced basis methods*, SIAM J. Math. Anal., **43** (3) (2011), 1457–1472.
3. A. Buffa, Y. Maday, A. T. Patera, C. Prudhomme, G. Turinici, *A Priori convergence of the greedy algorithm for the parameterized reduced basis*, Math. Model. Numer. Anal., **46** (2012), 595–603.
4. R. DeVore, G. Petrova, P. Wojtaszczyk, *Greedy algorithms for reduced bases in Banach spaces*, Constr. Approx., **37** (2013), 455–466.
5. P. Drabek, Y. Milota, *Methods of Nonlinear Analysis. Applications to Differential Equations*, Birkhäuser, Berlin, 2007.
6. P. Drabek, A. Kufner, F. Nicolosi, *Nonlinear Elliptic Equations, Singular and Degenerate Cases*, University of West Bohemia, 1996.
7. L. C. Evans, R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton, 1992.
8. R. Griesse, *Stability and Sensitivity Analysis in Optimal Control of Partial Differential Equations*, Karl-Franzens University Graz, Habilitation Thesis, 2007.
9. M. Choulli, E. Zuazua, *Lipschitz dependence of the coefficients on the resolvent and greedy approximation for scalar elliptic problems*, Comptes Rendus Mathématique, Elsevier Masson, **354** (2016), 1174–1187.

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