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TASK SOLVING ORGANIZATION OF THE INVERSE THERMOELASTICITY PROBLEM FOR A RECTANGULAR PLATES

The approach for solving the inverse problem of thermoelasticity, based on the method of the functions of influence, is proposed. The use of the functions of influence makes it possible to represent the temperature and the thermal voltage depending on the same desired vector. The numerical results of the identification of the thermal load measured with the error of thermal stress, which is characterized by a random quantity distributed under the normal law. The approach considered in this article is adapted to the problems of determining the non-stationary temperature and thermo-stressed states of isotropic two-layer hollow long cylinders and balls in the absence of information on the thermal load on one of the boundary surfaces. Under the well-known behavior in time of temperature and radial displacements of another boundary surface on the basis of the proposed method, problems are formulated, which are reduced to the inverse thermoelasticity problems, which are described by the Volterr integral equations of the first kind of convolution. Thermoelastic deformations have been discussed and illustrated numerically with the help of temperature and determined. The study of the kernels of the obtained integral equations for the considered bodies showed that they are additionally defined, monotonically increasing and have a root feature in the interval, i.e., they are Abel type integral equation. The functional spaces, for which the problems are well-posed, have been found. According to the results, obtained analytically, we can conclude, that the conditions for agreeing the values of the initial temperature, given radial displacements and pressures inside and outside the system at the initial moment of time are fulfilled. The basis of the model is the parametrization of a direct problem of nonlinear theory thin-walled elements using the boundary elements method г and the variational formulation of the identification problem, which provides for minimization of the residual functional reflecting the deviation of stress-strain state parameters obtained as a result of observationfrom those calculated on the basis of an approximate solution.

Keywords: *inverse problem, thermal stress, functions of influence, spline, identification, regularization, functional.*

1. Introduction

Experimental determination of quantities, which are included in mathematical models of thermal processes, in view of their complexity and imperfection often cannot serve as an exhaustive source of information on the conditions of unambiguousness. Lately in connection with this great attention is paid to the solution of inverse problems of thermal conductivity and thermoelasticity, in which according to the available (very limited) information about temperature voltages inside the body it is possible to determine thermophysical properties and geometric characteristics of the object. Also it is possible to identify the initial and boundary conditions, as well as clarify the mathematical model of the phenomenon itself. Such tasks can arise in remote measurements, under non destructive control of the state of structures, when studying thermal effects on descent spacecraft, in determining the thermophysical properties of new materials, etc.

Effectiveness of the decisions when designing various industrial equipment depends both on the depth and reliability of the knowledge of the phenomena of heat transfer, and from the adequacy of modeling thermophysical processes. The basis of the simulation methods, diagnostics and identification of processes Heat transfer can be made by solving inverse problems of thermal conductivity and thermoelasticity. In some cases, the methods for solving inverse problems are practically the only way to obtain the necessary information about the object under study.

The purpose of solving inverse problems of thermoelasticity (IPTE) may be, for example, an estimation of the temperature field according to the measurement of the thermal stress inside the body.

The methods for solving inverse problems make it possible to carry out research in the conditions that are as close as possible to full-scale, or directly during the exploitation of objects, which allows them to be more reasonably designed [1-3].

2. Statement of the problem

The plane stressed state is considered. The flat theory of elasticity is applied to the problem of analysis of thin rectangular plates, on which the load in the plane acts. Consider a thin elastic body, the thickness of which is very small in comparison with two other dimensions. The load is caused by mass forces b_x, b_y and marginal stresses σ_x, σ_y. Typically, it is assumed that the voltages are symmetrically distributed relatively to the average body plane, although more often the change in *h* thickness is considered constant. In this case, the final value of mechanical distress is not independent of *z*, but if the thickness of *h* is very small and it is assumed with sufficient degree of accuracy that

$$\sigma_z = 0, \tau_{xz} = 0, \tau_{yz} = 0$$

h then the remaining voltage components do not depend on the variable z, i.e.

$$\sigma_x = \sigma_x(x,y), \sigma_y = \sigma_y(x,y), \tau_{xy} = \tau_{xy}(x,y).$$

Thus, the defining equations have the form:

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y), \varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x), \gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy}. \quad (1)$$

The equilibrium equations are reduced to two, as in the case of flat deformation,

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + b_x = 0, \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + b_y = 0. \quad (2)$$

All equations for a plane stressed state can be obtained from the corresponding equations of plane deformation, if we use the actual elastic μ and E. Thus, we have: elastic steels

$$\mu = G = \frac{E}{2(!+\nu)}, \lambda^* = \frac{\nu E}{1-\nu^2}, \quad (3)$$

where λ^* – is a Lame constant; defining equations

$$\{\varepsilon\} = [S]\{\sigma\}, \{\sigma\} = [C]\{\varepsilon\},$$

$$[S] = \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}, \quad (4)$$

$$[C] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}. \quad (5)$$

Navier equilibrium equation

$$\begin{aligned} &\nabla^2 u + \frac{1+\nu}{1-\nu}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 \nu}{\partial x \partial y}\right) + \frac{1}{G}b_x = 0, \\ &\nabla^2 \nu + \frac{1+\nu}{1-\nu}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 \nu}{\partial x \partial y}\right) + \frac{1}{G}b_y = 0. \end{aligned} \quad (6)$$

Limit loads (forces)

$$\begin{aligned} &t_x = \lambda^*\left(\frac{\partial u}{\partial x} + \frac{\partial \nu}{\partial y}\right)n_x + \mu\left(\frac{\partial u}{\partial x}n_x + \frac{\partial \nu}{\partial x}n_y\right) + \mu\frac{\partial u}{\partial n}, \\ &t_e = \lambda^*\left(\frac{\partial u}{\partial x} + \frac{\partial \nu}{\partial y}\right)n_e + \mu\left(\frac{\partial u}{\partial x}n_x + \frac{\partial \nu}{\partial x}n_y\right) + \mu\frac{\partial \nu}{\partial n}. \end{aligned} \quad (7)$$

Initial stresses caused by temperature changes

$$\{\sigma_0\} = \frac{E\alpha\Delta T}{1-\nu}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}. \quad (8)$$

3. Methods

The thermoelastic bend of the hinged supported plates is considered. Thin plate deflection $w(x,y)$, which occupies two-dimensional area Ω in the plane xy, satisfies the equation [4]

$$\nabla^4 w = \frac{f}{D},$$

where $D = \frac{Eh^3}{12(1-\nu^2)}$ – bending stiffness of the plate;

ν – Poisson coefficient;

h – constant plate thickness;

$f = f(x,y)$ – distributed transverse load and

$$\nabla^4 = \nabla^2\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

– biharmonic operator.

Bending and torque points are given by expressions

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right);$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right); \qquad (9)$$

$$M_{xy} = -M_{yx} = D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}.$$

For the case of a hinged plate, the deflection must satisfy the following BC on the edge of the plate *Г*:

$$w = 0;$$

$$M_n = -D\left(\frac{\partial^2 w}{\partial n^2} + \nu\frac{\partial^2 w}{\partial t^2}\right) = 0, \qquad (10)$$

where M_n – bending moment in the direction *n* that is normal to the limit, *t* denotes tangent to the direction of the boundary.

Note that for the curvilinear limit

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial s^2} + k\frac{\partial w}{\partial n}, \qquad (11)$$

where $k = k(s)$ – curvature of the boundary.When the boundary of the hinged plate consists of straight lines, we have the following expressions

$$k = 0, w = 0, \frac{\partial w}{\partial s} = 0, \frac{\partial^2 w}{\partial s^2} = 0.$$

In this case, from the equation (11) we have

$$\frac{\partial^2 w}{\partial t^2} = 0,$$

but from the second equation (10)

$$\frac{\partial^2 w}{\partial n^2} = 0.$$

So, based on the above two equations, the deflection must satisfy the following equation at the boundary of the plate:

$$\nabla^2 w = \frac{\partial^2 w}{\partial n^2} + \frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0$$

on the border *Г*.

From equation (11) it follows that for points inside the domain Ω

$$M_x + M_y = -D(1+\nu)\nabla^2 w.$$

So, considering

$$M = \frac{M_x + M_y}{1+\nu} = -D\nabla^2 w \qquad (12)$$

one can rewrite the equation (12) in the form

$$\nabla^2(-D\nabla^2 w) = -f.$$

This equation can be divided into two potential equations:

$$\nabla^2 M = -f;$$
$$\nabla^2 w = -\frac{M}{D}. \qquad (13)$$

From equation (13) it follows that the boundary of the plate $M = 0$. Therefore, the solution of equation (13) for a hinged plate with a polygonal boundary can be obtained from two following Dirichlet problems:

$$\nabla^2 M = -f \text{ in } \Omega$$
$$M = 0 \text{ on } \mathit{Г}$$

$$\nabla^2 w = -\frac{M}{D} \text{ in } \Omega$$

$$w = 0 \text{ on } \mathit{Г}.$$

The solution of the equation of thermoelastic plate bending by dividing it into two potential equations belongs to Marcus [5,6]. Its use is limited, as this solution can only be applied to hinged-plate with polygonal boundary. The solution of the problem of deflection of a plate in the general case can be obtained by the variant MBC, developed for the biharmonic operator [7-9].

To solve equations (10) and (11) using MBC it is necessary to find integrals in the region

$$\int_\Omega \upsilon f d\Omega \text{ та } \int_\Omega \upsilon M d\Omega$$

where υ – the fundamental solution of the Laplace equation.

4. Results

A hinged square plate under the action of uniform loading; a square plate that is touched by the contour and is under the action of a uniform load; a square plate with a cutout; rectangular plate of two areas; a square composed plate have been considered and the calculation of folded plates was performed.

The hinged square plate has the following parameters:

$$a = 0,2 m; h = 0,001 m; E = 2\cdot 10^{10}\,Pa; \nu = 0,3.$$

The feasibility is to compare the solution obtained with the method of boundary elements (MBC) in conjunction with the variational method and the solution that can be considered accurate. A tendency was found as to the convergence of the solution, depending on the number of boundary elements (BC) and the order of approximation on these elements. Elements had equal length. Convergence is improved with increasing the order of approximation. It was found that given the size of the matrix of a separate system S, polynomial approximation is preferred.

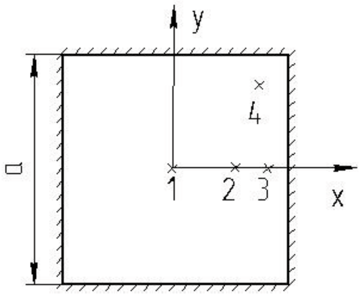

Fig. 1. Hinged square plate

A polynomial approximation was used on large fragment boundaries. Geometric boundary conditions (BC) are satisfied according to the scheme of the mean-square approximation. The temperature is defined by means of solution IPTE using the MBC.

Next, a square plate, which was clamped on the contour under the action of a uniform load, was considered. (Fig. 2).

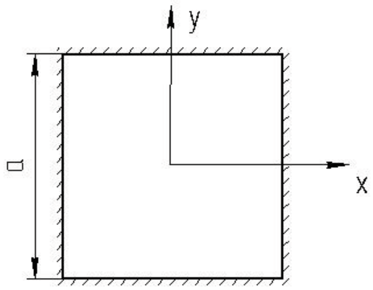

Fig. 2. Square plate that is pinched in the contour under the action of auniform load

The plate has the following parameters, as in the previous case:

$$a = 0,2m; h = 0,001m; E = 2 \cdot 10^{10} Pa; \nu = 0,3.$$

The results are presented in Fig.3.

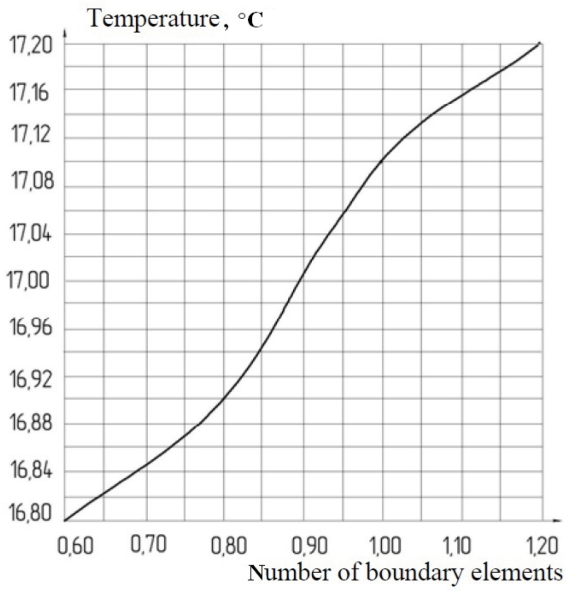

Fig. 3. Temperature distribution, depending on the number of boundary elements

This example is interesting because the solution is given at some points. Also, you can compare the results for the classical approximation scheme MBC and schemes of polynomial approximation on large fragment boundaries. Moreover, in the second case, for the complete solution of the problem there is no need to use the Lagrangian functional. Also, the temperature is defined with the help of solution IPTE using the MBC.

A square plate with a cutout is also considered (Fig. 4). The square hole is symmetrically placed. The outer contour is hinged, internal - free. The plate is evenly loaded. The parameters of the plate are the same as in other objects:

$$a = 0,2m; h = 0,001m; E = 2 \cdot 10^{10} Pa; \nu = 0,3.$$

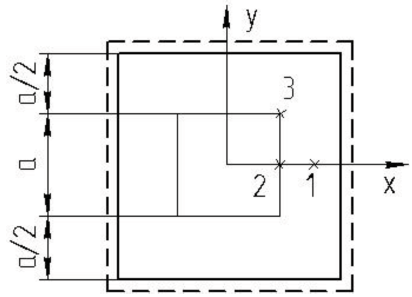

Fig. 4. Square plate with a cutout

In this case, the solution is obtained as in conjunction MBC with avariational approach, and in the version of the classical approach MBC. In the first case, 8 boundary fragments $(L = 8)$ and 6 – in the second case. Both approximations give similar results. With an increase *L* the results are practically unchanged, and for large ones *L* there are problems due to the great order of the system. In the second case, the contour is divided into 48 BC with quadratic approximation of compensating loads. On the free edge the contour in the area of the corner points are not fixed. There is a good match for the results with these approaches. The temperature is based on the solution IPTE using the MBC.

Also, a square composite plate was considered. The plate is evenly loaded. The coupling is performed on two sections. The parameters of the plate:

$$a = 0,2m; h = 0,001m; E = 2 \cdot 10^{10} Pa; \nu = 0,3.$$

As an initial reference point in this problem, the solution for a single square plate is given, which with an error of less than 1% coincides with a known solution. Then the results for that plate are shown, but made up of two sub-areas of the same rigidity. The following are variants when changing the thickness of the second sub-area. The temperature is based on the solution IPTE using the MBC.

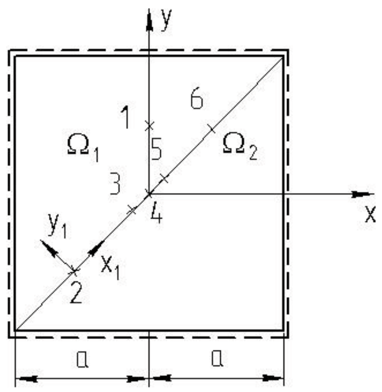

Fig. 5. Square composite plate

Also, a rectangular plate, consisting of two subareas, was considered. The coupling can be of elastic or hinge type. Parameters of the plate:

$$a = 0,2m; h = 0,001m; E = 2 \cdot 10^{10} Pa; \nu = 0,3.$$

This test allows you to evaluate the quality of the calculations.

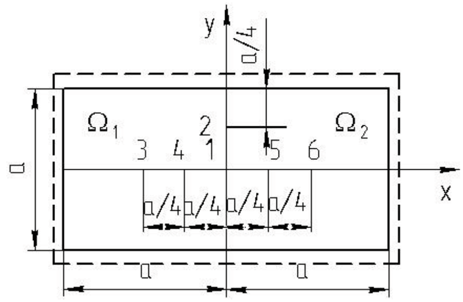

Fig. 6. Rectangular plate consisting of two areas

5. Conclusions

On the basis of algorithmic language FORTRAN a program for numerical calculations is developed. The first and easiest option is based on splitting the contour of the border by a broken line. Sections of this line are associated with BE. Within BE the approximation of compensating loads seems to be piecewise linear. With these provisions, the testing of programs with simple areas for which there are known solutions was performed. In addition, at this stage, mechanical BC could be realized hinged stop and pinching. When solving a test task with a circular plate, which was pinched in the contour, it was noted the difference of 10-12% by the value of the contour bending moment in comparison with the analytical solution. This error in the value of the contour bending moment is a consequence of replacing the contour line with a broken line. That is, it is important to take into account the curvature of the contour and, if in the quality BE are presented not straight, but corresponding arcs, the numerical solution practically coincides with the analytic. The next important point is the rate of convergence of the solution, depending on the quantity BE. Of course, the accuracy of the results obtained depends on the given accuracy of the calculation, as well as from the definition of compensating loads. It is determined that within the framework of the piecewise linear approximation, the accuracy of the solution depends directly on the increase in the number BE. The practical reception here is to partition the boundaries of a certain number of identical BE. For example, in a rectangular plate, each side is divided into a number of equal lengths BE. The accuracy of the solution at internal points is increasing rapidly with increasing the number of such BE. But when approaching the contour point, the rate of convergence of the solution slows down. This indicates that it is necessary to improve the quality of the approximation of compensating loads. With an increase in the number BE the quality situation in satisfaction should improve BC because the number of points of collocations is increased. But the increase in the number of points of collocation leads to an increase in the order of the system of linear equations and, accordingly, increases the rounding errors in its solution.

Test tasks confirm that starting from a certain point, the solution in the internal points gets worse, further increase in quantity BE becomes meaningless. Partial improving the solution can be due to uneven distribution BE on the contour (boundaries). For example, for rectangular plates, the quality of the solution improves with arelative decrease in length BE to the corner points. But similar studies are done to optimize both quantity and length BE are possible only for tasks with known solutions and they are quite laborious.Thus, the main issue when conducting numerical calculations is to develop certain criteria for the quality of the results. It is noted that the more accurately completed BC are, the more precise the solution is. At the points of collocation BC are performed exactly.

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В.О. Повгородний, О.С. Буданова. Организация решения обратных задач термоупругости для прямоугольных пластин

Новые обратные задачи термоупругости для прямоугольных пластин были сформулированы и применяются при проектировании устройств аэрокосмической техники. В этих задачах неизвестная тепловая нагрузка (температура граничной поверхности и интенсивность теплового потока) была определена с использованием данных вертикального смещения одной из внешних граничных поверхностей. Функциональные пространства, для которых обратные задачи корректны,были найдены. Способ решения обратных задач, *был предложен и проверен с использованием многократного решения прямой задачи. Эта статья посвящена определению температур нагрева и распределения температур на верхней поверхности тонкого кольца. Выражения температур нагрева и распределения температур были получены в виде ряда, включая функции Бесселя с помощью интегрального преобразования. Термоупругие деформации были обсуждены и проиллюстрированы численно с помощью численных методов определения температур.*

Ключевые слова: *обратная задача, обратная переходная функция, термоупругая деформация, прямоугольная пластина.*

В.О. Повгородній, О.С. Буданова. Організація вирішення обернених задач термопружності для прямокутних пластин

Нові обернені задачі термопружності для прямокутних пластин були сформульовані та використовуються при проектуванні пристроїв аерокосмічної техніки. В цих задачах невідоме теплове навантаження (температура граничної поверхні та інтенсивність теплового потоку) було визначене з використанням даних вертикального зміщення однієї з зовнішніх граничних поверхонь. Функціональні простори, для котрих обернені задачі коректні, були знайдені. Засіб використання обернених задач, був запропонований та перевірений з використанням багатократного вирішення прямої задачі.

Ця стаття присвячена визначенню температур нагріву та розподіленню температур на верхній поверхні тонкого кільця. Вираження температур нагріву та розподілення температур були одержані у вигляді ряду, враховуючи функції Беселя за допомогою інтегрального перетворення. Термопружні деформації були розглянуті та проілюстровані чисельно за допомогою чисельних методів визначення температур.

Ключові слова: *обернена задача, обернена перехідна функція, термопружна деформація, прямокутна пластина.*

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