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A HISTORY, THE MAIN MATHEMATICAL RESULTS AND APPLICATIONS FOR THE MATHEMATICS OF HARMONY

We give a survey on the history, the main mathematical results and applications of the Mathematics of Harmony as a new interdisciplinary direction of modern science. In its origins, this direction goes back to Euclid's "Elements." According to "Proclus hypothesis," the main goal of Euclid was to create a full geometric theory of Platonic solids, associated with the ancient conception of the "Universe Harmony." We consider the main periods in the development of the "Mathematics of Harmony" and its main mathematical results: algorithmic measurement theory, number systems with irrational bases and their applications in computer science, the hyperbolic Fibonacci functions, following from Binet's formulas, and the hyperbolic Fibonacci l-functions (l=1,2,3,...), following from Gazale's formulas, and their applications for hyperbolic geometry, in particular, for the solution of Hilbert's Fourth Problem.

Keywords: golden ratio, Pascal's triangle and Fibonacci numbers, Binet's formulas, Gazale's formulas, hyperbolic Fibonacci functions, number systems with irrational bases, Fibonacci l-numbers and "metallic proportions," Hilbert's Fourth Problem

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ИСТОРИЯ, РЕЗУЛЬТАТЫ ГЛАВНАЯ МАТЕМАТИЧЕСКИЕ И ПРИЛОЖЕНИЙ ДЛЯ МАТЕМАТИКИ ГАРМОНИИ

В статье изложена история развития, основные математические результаты и применения математики гармонии как нового междисциплинарного направления современной науки. В своих истоках, это направление восходит к «Началам» Евклида. Согласно "гипотезе Прокла", главная цель Евклида состояла в том, чтобы создать завершенную геометрическую теорию Платоновых тел, которые ассоциировались в древнегреескои науке с концепцией «Гармонии Мироздания». В статье рассматриваются основные периоды в развитии "Математики Гармонии" и ее основные математические результаты: алгоритмическая теория измерения, системы счисления с иррациональными основаниями и их применение в информатике, гиперболические функции Фибоначчи, вытекающие из формул Бине, и гиперболические l-функции Фибоначчи (l = 1,2,3, ...), вытекающие из формул Газале, и их приложения для гиперболической геометрии, в частности, для решения Четвертой Проблемы Гильберта.

***Ключевые слова**: золотое сечение, треугольник Паскаля, числа Фибоначчи, формулы Бине, формулы Газале, гиперболические функции Фибоначчи, системы счисления с иррациональными основаниями, l-числа Фибоначчи, "металлические пропорции", четвертая проблема Гильберта.*

Part I. A History of the Mathematics of Harmony and Numeral Systems with Irrational radices

1. Introduction

In 2009 the International Publishing House "World Scientific" has published the book: Alexey Stakhov. The Mathematics of Harmony. From Euclid to Contemporary Mathematics and Computer Science"[1] [1]

The American philosopher Professor Scott Olsen, author of the excellent book "The Golden Section: Nature's Greatest Secret" [2] made a huge assistance in writing the book [1] and its preparation for the publication.

The publication of these books is a reflection of one of the most important trend in the development of modern science. The essence of this trend is very simple: a return to the "harmonic ideas" of Pythagoras and Plato (the "golden ratio" and Platonic solids), embodied in Euclid's "Elements" [3].

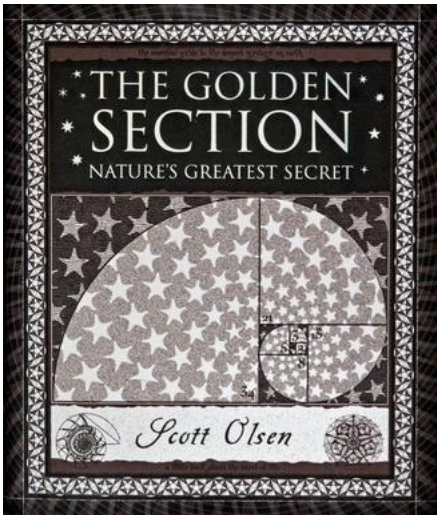
The newest discoveries in chemistry and crystallography: **fullerenes**, based on the "truncated icosahedron" (The Nobel Prize - 1996), and **quasi-crystals**, based on the icosahedral or pentagonal symmetry (Nobel Prize - 2011), are brilliant examples, which confirm this trend.

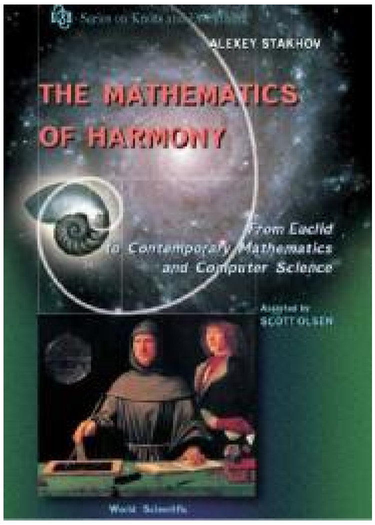
The amount of such discoveries is increasing continuously. These include: *"the law of structural harmony of systems"* by Edward Soroko [4], based on the golden ***p***-proportions, the *"law of spiral biosimmetry transformation"* by Oleg Bodnar [5], based on the "golden" Fibonacci hyperbolic functions. It also includes a new theory of the genetic code, based on the *"golden genomatrices"* (author - Doctor of Physical and Mathematical Sciences Sergey Petoukhov, Moscow) [6]. These examples could go on.

Thus, the modern philosophy and theoretical natural sciences begun to use widely the "harmonic ideas" by Pythagoras, Plato and Euclid. And we have every right to talk about the "revival" of ancient Greeks' "harmonic ideas" in modern theoretical natural sciences. This fact puts forward a problem of the renaissance of these ancient harmonic ideas in modern mathematics. The publication of the book [1] is the answer of mathematics to this important trend.

The purpose of this article is to give a survey of the main stages, events and scientific findings, which led to the creation of "Mathematics of Harmony," the new interdisciplinary direction of modern science.

[1] http://www.worldscientific.com/worldscibooks/10.1142/6635





Scott Olsen's book (2006) and Alexey Stakhov's book (2009)

2. The Mathematics of Harmony: the opinion of Academician Yuri Mitropolsky

What is the Mathematics of Harmony"? What is its role in modern science and mathematics? The outstanding Ukrainian mathematician, the leader of the Ukrainian School of Mathematics, Honorary Director of the Institute of Mathematics of the Ukrainian Academy of Sciences and Editor-in Chief of the "Ukrainian Mathematical Journal" Academician Yuri Mitropolsky wrote the following in his commentary [7]:

Academician Yuri Mitropolsky (1917 - 2008)

"One may wonder what place in the general theory of mathematics is occupied by Mathematics of Harmony created by Prof. Stakhov? It seems to me, that in the last centuries, as Nikolay Lobachevsky said, "mathematicians turned all their attention to the Advanced Parts of Analytics, neglecting the origins of Mathematics and not willing to dig the field that already been harvested by them and left behind." As a result, this created a gap between "Elementary Mathematics" - basis of modern mathematical education, and "Advanced Mathematics". In my opinion, the Mathematics of Harmony developed by Prof. Stakhov fills up that gap. I.e, "Mathematics of Harmony" is a big theoretical contribution, first of all to the development of "Elementary Mathematics" and as such should be considered of great importance for mathematical education."

Thus, according to Mitropolsky, the Mathematics of Harmony is a new mathematical discipline, which fills the gap between the "Elementary Mathematics" and "Higher Mathematics. " That is, this new theory puts forward new challenges in the field of the "Elementary Mathematics." In its origins, this theory goes back to the ancient mathematical topics: "Measurement Theory," "Number Theory," "Numeral Systems," "Elementary Functions " and so on.

The main task of the "Mathematics of Harmony" is to find new mathematical results in the field of the "Elementary Mathematics" based on the "golden ratio" and Fibonacci numbers. It is proved a high efficiency of these results in such areas as theory of recurrence relations, theory of elementary functions, hyperbolic geometry, and finally, the computer and measuring engineering, coding theory [8-41].

3. About the term of "the Mathematics of Harmony"

For the first time the term "the Mathematics of Harmony" was used in the article "Harmony of spheres", placed in «The Oxford dictionary of philosophy» [42]:

***"Harmony of spheres.** A doctrine often traced to Pythagoras and fusing together mathematics, music, and astronomy. In essence the heavenly bodies being large objects in motion, must produce music. The perfection of the celestial world requires that this music be harmonious, it is hidden from our ears only because it is always present. The mathematics of harmony was a central discovery of immense significance to the Pythagoreans."*

Thus, the concept of "the Mathematics of Harmony" is associated here with the "harmony of the spheres", which is also called the "harmony of the world» (Latin "harmonica mundi") or world music (Latin "musical mundane"). The harmony of the spheres is the ancient and medieval doctrine about the musical-mathematical construction of Cosmos, which goes back to the Pythagorean and Platonic philosophical tradition.

Another mention about "the Mathematics of Harmony," as the ancient Greek great discovery, we find in the book by Vladimir Dimitrov. "A new kind of social science. Study of self-organization of human dynamics [43]. Let us consider the quote from the book:

"*Harmony was a key concept of the Greeks, a conjunction of three strands of meaning. Its root meaning* *was **aro**, join, so **harmonia** was what joined. Another meaning was proportion, the balance of things that allowed an easy fit. The quality of joining and proportion then came to be seen in music and other arts.*

The precondition for harmony for the Greeks was expressed in the phrase "nothing to much". It also had a mysterious positive quality, which became the object of enquiry of their finest minds. Thinkers such as Pythagoras sought to capture the mystery of harmony as something both inexpressible yet also illuminated by mathematics. The mathematics of harmony explored by the ancient Greeks is still an inspiring model for contemporary scientists. Crucial to it is their discovery of its quantitative expression in astonishing diversity and complexity of nature through the golden mean (golden ratio), Φ (phi): $\Phi = \frac{1+\sqrt{5}}{2}$, *which is approximately equal to 1.618. It is described by Euclid in book five of his Elements:"A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to greater, so is greater to the less".*

Thus, in the book [43] the concept of "the Mathematics of Harmony" is directly associated with the "golden ratio" - the most important ancient mathematical discovery in the field of harmony, which at that time was called "the division of the segment in extreme and mean ratio."

Finally, it is pertinent to mention that this term was used in Stakhov's speech «The Golden Section and Modern Harmony Mathematics», made at the 7th International Conference «Fibonacci Numbers and Their Applications" (Austria, Graz, 1996) [11].

4. The most important periods in the development of the Mathematics of Harmony

As mentioned above, very successful term of the "Mathematics of Harmony" was introduced to emphasize the most important feature of ancient Greek science (studying the Harmony of the Universe) [42, 43]. The greatest interest in the "Harmony of Universe", that is, in the ideas of Pythagoras, Plato and Euclid, always arose in the periods of greatest prosperity of the "human spirit." From this point of view, the studying the "Mathematics of Harmony" can be divided into the following critical periods.

4.1. Ancient Greek period

Conventionally, it can be assumed that this period starts with the research of Pythagoras and Plato. Euclidean' "Elements" is a final event of this period. According to Proclus' hypothesis [3], Euclid created his "Elements" in order to give a complete geometric theory of the five "Platonic solids," which have been associated in the ancient Greek science with the "Harmony of the Universe." Fundamentals of the theory Platonic solids (Fig.1) were placed by Euclid in the final Book (Book XIII) of his "Elements."

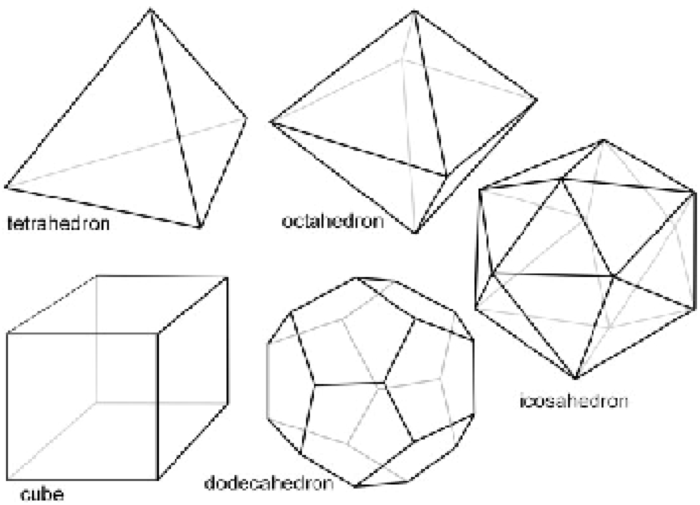


Figure 1. Platonic solids: tetrahedron, octahedron, cube, dodecahedron, icosahedron

In addition, Euclid simultaneously had introduced here some advanced achievements of ancient Greek mathematics, in particular, the "golden ratio" (Book II), which was used by Euclid for the creation of the geometric theory of "Platonic solids.

4.2. The Middle Ages

In the Middle Ages it was made very important mathematical discovery. The famous Italian mathematician Leonardo of Pisa (Fibonacci) wrote a book «Liber Abaci" (1202). In this book, he described "the task of rabbits reproduction." In solving this problem he introduced the remarkable numerical sequence - the Fibonacci numbers F_n: 1,1,2,3,5,8,13,21,34,55,89,..., which are given by the recurrence relation:

$$F_n = F_{n-1} + F_{n-2}; \quad F_1 = F_2 = 1 \tag{1}$$

The Fibonacci numbers can be extended to the negative values of the indices n (see Table 1).

Table 1
The “extended” Fibonacci numbers

n	0	1	2	3	4	5	6	7	8	9	10
F_n	0	1	1	2	3	5	8	13	21	34	55
F_{-n}	0	1	–1	2	–3	5	–8	13	–21	34	–55

In 17 c. the famous French astronomer Cassini had proved the following remarkable formula, which connects three neighboring Fibonacci numbers:

$$F_n^2 - F_{n-1}F_{n+1} = (-1)^{n+1} \tag{2}$$

4.3. The Renaissance

This period is connected with the names of the prominent figures of the Renaissance: Piero della Francesca (1412-1492), Leon Battista Alberti (1404-1472), Leonardo da Vinci (1452-1519), Luca Pacioli (1445-1517), Johannes Kepler (1571-1630). In that period two books, which were the best reflection of the idea of the "Universe Harmony," were published. The first of them is the book "Divina Proprtione" («The Divine Proportion") (1509). This book had been written by the outstanding Italian mathematician and scholar monk Luca Pacioli under the direct influence of Leonardo da Vinci, who illustrated Pacioli’s book. The brilliant astronomer of 17th century Johannes Kepler made an enormous contribution to the development of the "harmonic ideas" of Pythagoras, Plato and Euclid.

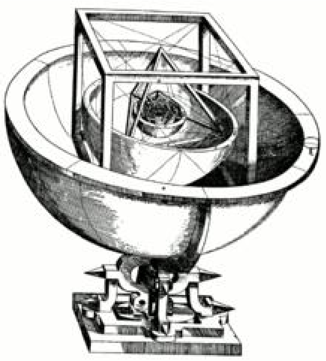


Figure 2. Kepler’s *Cosmic Cup*

In his first book ***Mysterium Cosmographicum*** (1596) he built the so-called *“Cosmic Cup”* (Fig.2) - an original model of the Solar system, based on the Platonic solids. The book *Harmonice Mundi* (Harmony of the Worlds) (1619) is the main Kepler’s contribution into the Doctrine of the Universe Harmony. In *Harmony*, he attempted to explain the proportions of the natural world—particularly the astronomical and astrological aspects—in terms of music. The central set of "harmonies" was the *musica universalis* or "*music of the spheres*," which had been studied by Pythagoras, Ptolemy and many others before Kepler.

4.4. The 19th century

This period is connected with the works of the French mathematicians Jacques Philip Marie Binet (1786-1856), Francois Edouard Anatole Lucas (1842-1891), German poet and philosopher Adolf Zeising (born in 1810) and the German mathematician Felix Klein (1849 - 1925).

Jacques Philip Marie Binet derived a mathematical formula to expresses the "extended" Fibonacci numbers (see Table 1) through the "golden mean" $\Phi = \frac{1+\sqrt{5}}{2}$:

$$F_n = \begin{cases} \dfrac{\Phi^n + \Phi^{-n}}{\sqrt{5}} & \text{for} \quad n = 2k+1; \\ \dfrac{\Phi^n - \Phi^{-n}}{\sqrt{5}} & \text{for} \quad n = 2k \end{cases} \tag{3}$$

where k=0,±1,±2,±3,….

Francois Edouard Anatole Lucas introduced the Lucas numbers L_n, which are calculated by the same recurrence relation as the Fibonacci numbers (2), but with other seeds:

$$L_n = L_{n-1} + L_{n-2}; \quad L_1 = 1, L_2 = 3 \tag{4}$$

Recurrence relation (4) generates the following numerical sequence:

$$L_n : 1, 3, 4, 7, 11, 18, 29, ... \tag{5}$$

The Lucas numbers (5) can be extended to the negative values of the indices n (see Table 2).

Table 2
The “extended” Lucas numbers

n	0	1	2	3	4	5	6	7	8	9	10
L_n	2	1	3	4	7	11	18	29	47	76	123
L_{-n}	2	–1	3	–4	7	–11	18	–29	47	–76	123

The merit of Binet and Lukas is the fact that their researches became a launching pad for Fibonacci researches in the Soviet Union, the United States, Britain and other countries [44-46].

German poet Adolf Zeising in 1854 published the book *«Neue Lehre von den Proportionen des menschlichen Körpers aus einem bisher unerkannt gebliebenen, die ganze Natur und Kunst durchdringenden morphologischen Grundgesetze entwickelt»*. The basic Zeising's idea is to formulate the Law of proportionality. He formulated this Law as follows:

"A division of the whole into unequal parts is proportional, when the ratio between the parts is the same as the ratio of the bigger part to the whole, this ratio is equal to the golden mean".

The famous German mathematician Felix Klein in 1984 published the book *"Lectures on the icosahedron and the solution of equations of the fifth degree,"* dedicated to the geometric theory of the icosahedron and its role in the general theory of mathematics. Klein treats the icosahedron as a mathematical object, which is a source for the five mathematical theories: *geometry, Galois theory, group theory, invariant theory and differential equations.*

What is the significance of the ideas of the outstanding mathematician from the point of view of the Mathematics of Harmony [1]? According to Klein, the Platonic icosahedron, based on the "golden ratio," is the main geometric figure of mathematics. It follows from this that the "golden ratio" is the main geometric object of mathematics, which, according to Klein, can unite all mathematics.

This idea of Felix Klein is consistent with the ideas of the article "Generalized Golden Sections and a new approach to geometric definition of a number," published by the author in the "Ukrainian Mathematical Journal" [24]. This article presents the concept of the "golden" number theory, which can be the basis for the "golden" mathematics, based on the "golden ratio" and its generalizations.

4.5. The first half of the 20th century

In the first half of the 20th century the development of the "golden" paradigm of the ancient Greeks is associated with the names of the Russian Professor of architecture Grimm (1865-1942) and the classic of the Russian religious philosophy Paul Florensky (1882-1937).

In the theory of architecture, it is well-known the book "Proportionality in Architecture", published by Prof. Grimm in 1935 [47]. The purpose of the book has been formulated in the "Introduction" as follows:

"In view of the exceptional significance of the Golden Section in the sense of the proportional division , which establishes a permanent connection between the whole and its parts and gives a constant ratio between them (which is unreachable by any other division), the scheme, based on it, is the main standard and is accepted by us in the future as a basis for checking the proportionality of historical monuments and modern buildings ... Taking this general importance of the Golden Section in all aspects of architectural thought , the theory of proportionality, based on the proportional division of the whole into parts corresponding to the Golden Section, should be recognized as the architectural basis of proportionality at all."

In the 20th years of 20th century, Pavel Florensky wrote the work "At the watershed of a thought." Its third chapter is devoted to the "golden ratio". The Belorussian philosopher Edward Soroko in the book [4] evaluates Florensky's work as follows:

" *The aim was to derive analytically the stability of the whole object , which is in the field of the effect of oppositely oriented forces. The project was conceived as an attempt to use the "golden ratio and its substantial basis, which manifests itself not only in a series of experimental observations of nature, but on the deeper levels of knowledge, for the case of penetration into the dialectic of movement, into the substance of things."*

4.6. The second half of the 20th century and the 21st century

In the second half of 20^{th} century the interest in this area is increasing in all areas of science, including mathematics. The Soviet mathematician Nikolai Vorobyov (1925-1995) [44], the American mathematician Verner Hoggatt (1921-1981) [45], the English mathematician Stefan Vajda [46] and others became the most outstanding representatives of this trend in mathematics.

Reviving the idea of harmony in modern science is determined by new scientific realities. The penetration of the Platonic solids, the "golden ratio" and Fibonacci numbers in all areas of theoretical natural sciences (crystallography, chemistry, astronomy, earth science, quantum physics, botany, biology, geology, medicine, genetics, etc.), as well as in computer science and economics was the main reason for the renewed interest in the ancient idea of the Universe Harmony in modern science and the stimulus for the development of the "Mathematics of Harmony" [1–41].

4.7. The article by Victor Shenyagin

Recently on the website "Academy of Trinitarizm, Institute of the Golden Section" has published the article by well-known Russian Fibonacci-scientist Victor Shenyagin [48] with the following appeal:

"Appeal to the international scientific community: On the appropriateness of nomination of Professor Stakhova AP to award the Abel Prize in Mathematics in 2014"

Of course, it is pleasant for the author to get a high appreciation of his research from his colleagues. Similar estimations were given also by other well-known scientists: the Ukrainian mathematician academician Yuri Mitropolsky [7], the American philosopher Scott Olsen [49], the Belarusian philosopher Edward Soroko [50], the Russian philosopher Sergei Abachiev [51], and many others. Their feedback is the highest award for the author.

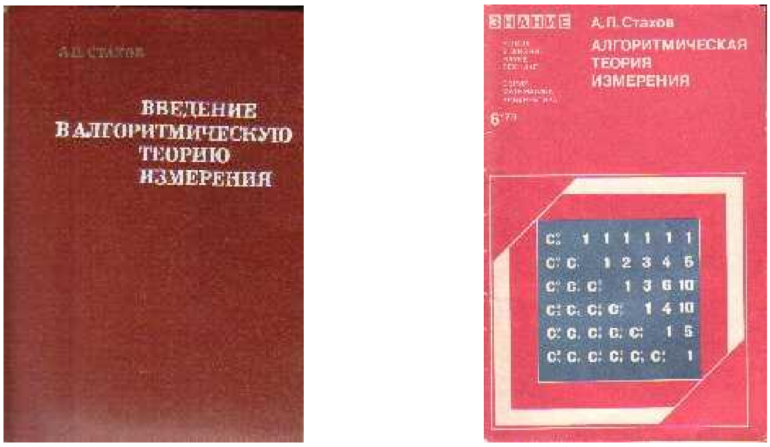
Unfortunately, this scientific discipline, called the "Mathematics of Harmony" [1] until now is not a generally recognized part of the conventional mathematics. However, this scientific direction became widely known in modern science after the publication of the book [1], as well as the author's publications in the Ukrainian academic and English scientific journals [3, 8-41].

Let us consider the basic mathematical results and theories of the "Mathematics of Harmony," mentioned by Victor Shenyagin in the article [48].

5. Algorithmic measurement theory

The algorithmic measurement theory is the first mathematical theory, created by the author during 70th years of 20 c. The foundations of this theory are described in the books [12, 13].

Although this theory by its initial idea had purely applied character ("Synthesis of the optimal algorithms for analog-to-digital conversion", by the way, this was the title of author's doctoral dissertation, 1972), but in process of creating this theory, the author went out far for the application frameworks and touched the foundations of mathematics.



Stakhov books on the Algorithmic Measurement Theory (1977, 1979) (in Russian)

First of all, the author have excluded from consideration the abstraction of *actual infinity* as an internally contradictory concept (the "completed infinity"), because still Aristotle protested against this. As a result, the author has excluded from the algorithmic measurement theory *Cantor's axiom*, which is the basis of the classical measurement theory, based on the *continuity axioms*. The author found the contradiction between Cantor's axiom and the Archimedes' axiom [12]. This contradiction implies automatically the contradictions in the foundations of mathematics. The famous Russian mathematician and philosopher Alexander Zenkin had developed these ideas in the article [52], where he criticizes Cantor's theory of infinite sets.

Such a constructive approach to the algorithmic measurement theory, based on the abstraction of potential infinity, allowed the author to solve the problem, which was never been considered in mathematics. We talk about the problem of the synthesis of optimal measurement algorithms, which are a generalization of well-known measurement algorithms: the "counting algorithm," which lies at the basis of Euclidean definition of natural numbers, and the "binary algorithm," which lies at the basis of the binary system, the basis of modern computers.

Each measurement algorithm corresponds to some positional number system. From the algorithmic measurement theory, it follows all the known positional number systems. It is very important, that the new, unknown positional number systems, in particular, the "Fibonacci" and "binomial" number systems arose in the algorithmic measurement theory [12].

Thus, the algorithmic measurement theory is a source for the new positional number systems. Note that mathematics was never seriously engaged in number systems. The mathematics moved forward not very far in this area compared with the period of its origin. And this is the mistake of mathematics. New positional number systems (in particular, introduced by the author Fibonacci p-codes) can become an alternative to the binary system for many critical applications.

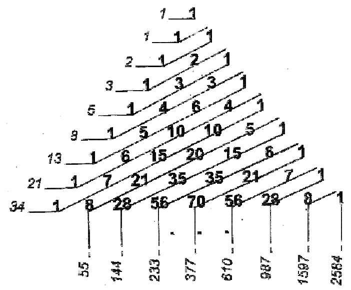


Figure 3. Pascal's Triangle and Fibonacci numbers

6. Pascal's Triangle and the generalized Fibonacci numbers

Pascal's Triangle is so widely known and studied mathematical object that many mathematicians will be surprised what's new can be found in this triangle? The book "Mathematical Discovery" (Russian translation, 1970) [53] by the American mathematician Polya is well-known. In this book Polya showed an unexpected connection of Pascal's triangle with the Fibonacci numbers.

The study of the so-called "diagonal sums" of Pascal's triangle, conducted by many mathematicians (including the author of this article), has led to the discovery of an infinite amount of the recurrent sequences, called the Fibonacci ***p***-numbers [12]. These recurrent sequences for the given $p = 0,1,2,3,...$ are generated by the following general recurrence relation:

$$F_p(n) = F_p(n-1) + F_p(n-p-1) \text{ для } n > p+1 \quad (6)$$

at the seeds:

$$F_p(1) = F_p(2) = ... = F_p(p+1) = 1. \quad (7)$$

7. A generalization of the "golden ratio"

By studying the Fibonacci ***p***-numbers and considering the limit of the ratio of neighboring Fibonacci ***p***-numbers $\lim_{n\to\infty} \frac{F_p(n)}{F_p(n-1)} = x$, the author came to the following algebraic equation [12]:

$$x^{p+1} - x^p - 1 = 0 \text{ } (p=0,1,2,3,..). \quad (8)$$

The positive roots of the equation (8) form a set of the new mathematical constants Φ_p, which describe some algebraic properties of Pascal's triangle. The classical "golden ratio" is a special case (*p*=1) of the constants Φ_p. On this basis, the constants Φ_p have been called the "golden *p*-proportions." This result has caused an admiration by academician Mitropolsky, who in his commentary [7] wrote the following:

"Let's ponder upon this result. Within several millennia, starting since Pythagoras and Plato, the mankind used the widely known classical Golden Proportion as some unique number. And here in the end of the 20th century the Ukrainian scientist Stakhov generalized this result and proved the existence of infinite number of the Golden Proportions! And all of them have the same right to express Harmony, as well as the classical Golden Proportion. Moreover, Stakhov proved, that the golden p-proportions $\Phi_p \left(1 \le \Phi_p \le 2\right)$ represented a new class of irrational numbers, which express some unknown mathematical properties of Pascal triangle. Clearly, such mathematical result is of fundamental importance for the development of modern science and mathematics."

8. The redundant binary positional number systems (the Fibonacci *p*-codes)

Fibonacci measurement algorithms, based on the Fibonacci ***p***-numbers [12], led to the discovery of the new positional number systems, called the Fibonacci ***p***-codes:

$$N = a_n F_p(n) + a_{n-1} F_p(n-1) + ... + a_i F_p(i) + ... + a_n F_p(1), \quad (9)$$

where *p*=0,1,2,3,... is a given integer; $a_i \in \{0,1\}$ is the binary numeral of the i th digit; the Fibonacci ***p***-number $F_p(i)(i = 1,2,3,...,n)$ is the weight of the i th digit.

This expression is a generalization of the classical binary code (*p*=0):

$$N = a_n 2^{n-1} + a_{n-1} 2^{n-2} + ... + a_i 2^{i-1} + ... + a_1 2^0, \quad (10)$$

which underlies the modern computers. For the case *p*>0 all the Fibonacci *p*-codes are redundant positional binary systems, which can be used for the design of new computers (Fibonacci computers), which have super-high informational reliability.

9. Foreign patents on the Fibonacci computers

In 1976, the author worked during 2 months as Visiting-Professor of the Vienna University of Technology. At the final stage of the stay in Austria, the author made a speech "Algorithmic Measurement Theory and Foundations of Computer Arithmetic" at the joint meeting of the Austrian Computer and Cybernetics Societies. The success of the speech was stunning. It was recognized that this speech provides new informational and arithmetical foundation of computers.

This sudden interest of Austrian scientists in author's speech became a reason of the letter of the Soviet Ambassador in Austria Ivan Yefremov, who sent it to the USSR State Committee for Science and Technology. On the basis of this letter, the Soviet State Invention Committee took the decision about the urgent patenting author's inventions on "Fibonacci computers " abroad. 65 foreign patents of the U.S. Japan, England, France, Germany, Canada and other countries are official legal documents, which confirmed a priority of the author in this area.

10. Codes of the Golden Proportion

In 1957, the young (12 year old) American wunderkind George Bergman in one of the U.S. mathematical Journals had described the unique number system, called a number system with an irrational base [54]:

$$A=\sum_i a_i\Phi^i\ , \qquad (11)$$

where A is a some real number, $a_i\in\{0,1\}$ is a binary numeral of the i th digit $(i=0,\pm1,\pm2,\pm3,...)$, Φ^i is the weight of the i th digit, $\Phi=\frac{1+\sqrt{5}}{2}$ is a base of “Bergman’s system (11). Unfortunately, the mathematicians of 20th century did not pay attention to this mathematical discovery, despite the fact that "Bergman’s system" (11) is the greatest contemporary mathematical discovery in the field of positional number systems after the discovery of the Babylonian positional principle of number representation and decimal and binary number systems.

In 1980, the author had generalized the number system (11) and introduced in mathematics a wide class of the positional number systems with irrational bases, called "codes of the golden *p*-proportions" [10]:

$$A=\sum_i a_i\Phi_p^i\ , \qquad (12)$$

where $i=0,\pm1,\pm2,\pm3,...,\ p=0,1,2,3,...,\ a_i\in\{0,1\}$ is a binary numeral of the i th digit, Φ_p is a base of the number system (12) (the golden *p*-proportion, following from Pascal’s triangle), Φ_p^i is the weight of the i th digit, connected with weights of the previous digits by the following identity:

$$\Phi_p^i=\Phi_p^{i-1}+\Phi_p^{i-p-1}\,. \qquad (13)$$

Theory of the codes of the golden *p*-proportions was set forth in the book [14]. This book attracted the attention of the Soviet scientific- popular Journal "Technology for young people." In 1985 this Journal had published author’s article "Codes of the Golden Proportion, or the number systems for future computers?" [55]. This article was the “key” article of this Journal what had been emphasized by the collage "Codes of the Golden Proportion," placed on the back cover of this Journal.

Stakhov’s book “Codes of the Golden Proportion” (1984) and the Journal "Technology for Young People" (1985)

The publication of author’s article in the famous scientific-popular Journal by circulation of 1.7 million copies caused a big interest of scientific community in this research direction, which has became well known not only in the USSR but also abroad.

Note that the expression (12) includes an infinite number of the positional binary number systems, because every $p\,(p=0,1,2,3,...)$ generates its own number system. In particular, for the case $p=0$ the base $\Phi_p=\Phi_0=2$ and the code of the golden *p*-proportion (12) is reduced to the classical binary system, and for the case $p=1$ to Bergman’s system (11).

11. The "golden" number theory

Academician Yuri Mitropolsky, Chief-Editor of the "Ukrainian Mathematical Journal," has invited the author to submit the article on the codes of the golden *p*-proportions for the Journal. In 2004, according to Mitropolsky’s recommendation, the article "The generalized golden proportions and a new approach to geometric definition of a number" was published in this prestigious mathematical Journal [24].

The main idea of the article is reduced to the following. The codes of the golden *p*-proportions, which are connected with Pascal's triangle, can be considered as the beginning of the new number theory, the "golden" number theory. Indeed, with the help of the codes of the golden *p*-proportions we can represent all real numbers, including natural, rational and irrational. The codes of the golden p-proportions change our ideas about the relationship between rational and irrational numbers, because the special irrational numbers (the golden p-proportions) are becoming the basis of all numbers and, therefore, of all mathematics.

Consider one unusual result of the "golden" number theory on the example of the Bergman's system (11). For this, we represent the natural number N in the Bergman's system (11):

$$N = \sum_i a_i \Phi^i . \tag{14}$$

It is proved [24] that the sum of (14) for any natural number N is finite always, that is, any natural number N can be represented as the finite sum of the powers of the "golden proportion". Since all powers of the "golden proportion" are irrational numbers (except for $\Phi^0 = 1$), then this assertion is far from trivial.

Let us consider the so-called "extended" Fibonacci numbers (see Table 1).

We substitute now the "extended" Fibonacci numbers $F_i\left(i = 0, \pm1, \pm2, \pm3, ...\right)$ instead the powers Φ^i into the expression (14). To our surprise, we find [24] that this sum is equal to 0 for any natural number N, that is,

$$\sum_i a_i F_i = 0. \tag{15}$$

This property has been called the Z-property of natural numbers [24]. Since this property is valid only for natural numbers, this means that in [24] we found a new property of natural numbers, via 2.5 thousand years of their theoretical study (academician Mitropolsky was thrilled!).

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