rail car is used into the body of which the construction of the device is arranged.

Keywords: *non-automatic weighing calibration device, load carrier, weight-transmitting device, weighting bridges, weightless verification, calibration of large-load scales, standard weight measuring instrument.*

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CONTACT PROBLEM OF COMBINED FOOTING RESTING ON ELASTIC BASE

The contact problem of combined footing resting on elastic half space, or stratified elastic base has been analyzed. Unlike simplified common assumptions and design methods of such foundation, here the analysis was carried over a two rigid footings connected by elastic flexible foundation beam, taking into account different characteristics of individual footing parts (rigid vs. flexible) and continuity conditions at their joints, with emphasis on their mutual interaction related to underlying soil load transfer. The numerical analysis has been performed by combined finite difference and boundary element method on the identical quadrilateral mesh.

Keywords: *combined footing, interaction, elastic half space, layered elastic medium, FEM, BEM, settlemet.*

Introduction.

Combined footings are common form of shallow foundations. A system of two rigid footings connected by a foundation beam are often referred as strap footing, but this term implies a special case of combined footing loading, which is exclusively used for compensation of highly eccentric concentrated loads [1], [2]. However, there are many other situations where the combined footings consisting of foundation beams and rigid footings are applicable, and actually used, and they are subject of this analysis in some broader sense.

Traditionally the use of the of shallow foundations is principally restricted to the relatively light structures due to limitations that are mainly related to two main reasons: the first being restrictions related to the shallow soil properties, their intrinsic non - homogenous nature and complex stress - strain behavior, and the second lack of correct assessment of foundation structure alone and soil - structure interaction effects, which makes predictions of real structure behavior unreliable and consequently can result in their inadequate design [3].

There are many publications considering various analytical and numerical methods and solutions for contact problems of simple-shaped foundations like flexible foundation beams, rigid or flexible plates of basic shapes. Some of them are reviewed and discussed in [4], [7] monographs [8], [9] and more related to the presented problem [10], [12] as well as their experimental verification [13]. But there is a lack of those deliberated to the more specific ones such as combined foundations, due to inherent complexity of any thorough analysis of such problems due to lack of accurate, i. e. closed-form analytical solutions for any but the simplest foundation shapes.

Consequently, in standard geotechnical engineering literature and engineering practice combined footings are treated on simplified way and designed almost solely on the basis of bearing capacity analysis, rather than considering influence of settlements and soil structure interaction effects.

As an example, in case of strap footings, in current design and construction practice the influence of beam-soil interaction is disregarded, and such possibility purposely prevented for compliance with design assumptions, and such approach is embedded in current design codes [14].

In common engineering practice this approach is not limited to the mentioned case, but it is widely used in most cases of combined beams with rigid footings on deformable bases.

Flexural rigidity of the structure can have significant influence on distribution of load and moments transmitted to the foundation of the structure, and the load redistribution may modify the pattern of or mitigate settlement [15].

So for safe and rational application of combined footing systems, better understanding their behavior, it is necessary to provide estimation of it's load distribution and resulting contact stresses and settlements [16].

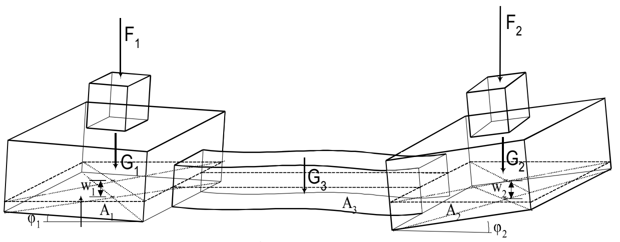
In this paper to a numerical analysis procedure of combined footing resting on elastic base has been presented, taking into account different characteristics of individual footing parts (rigid vs. flexible) with emphasis on their mutual interaction based on continuity conditions at their joints.

To determine a more realistic contact stresses distribution on the contact plane and consequently the settlements more accurately, the entire footing system has to be treated as a unique boundary value problem [17].

Statement of the problem.

On Fig. 1 the contact problem of combined footing consisting of two rigid footings connected by elastic flexible foundation beam is presented schematically. It is considered that footing system is resting on the soil surface modeled as linearly deformable isotropic elastic half - space or alternatively can be treated as elastic layered soil medium of finite depth. The analysis of the contact problem here was carried with special emphasis on their mutual interaction while transferring load to underlying soil.

The contact between the soil and the footing system is assumed frictionless. It is assumed that bends of flexible beam are small and the Euler-Bernoulli hypothesis is valid.



$$P_1 = \int_{A_1} p_1(x,y)dxdy \qquad P_3 = \int_{A_3} p_3(x,y)dxdy \qquad P_2 = \int_{A_2} p_2(x,y)dxdy$$

$$P_1 \neq (F_1 + G_1) \qquad P_3 \neq (F_3 + G_3) \qquad P_2 \neq (F_2 + G_2)$$

$$\sum_{n=1}^{N} P_n = \sum_{n=1}^{N} (F_n + G_n)$$

Figure 1 – Schematic representation the strip footing system: F – external forces, A – area, $p(x, y)$ – contact pressure, G – self weight sum forces, P – partial contact sum forces on respective footing parts, w and φ footing centre vertical displacements of and footing inclinations.

Although the eccentric application of load in relation to the longitudinal plane of symmetry or/and non-uniformity of soil properties can finally result in asymmetrical shape of deformed footing system, according to chosen theory and for reasons of clarity, this analysis is limited here on the simpler case, where it will be assumed that the system is loaded symmetrically relative to the longitudinal plane of symmetry of footing system.

The contact problem solution for the strip footing system is expected to satisfy requirements of continuity of displacement and load transfer at their joints in conditions of their mutual interaction.

Since behavior of construction parts has to be described by different behaviour concepts (rigid behaviour of footings and flexible behaviour of the interconnecting foundation beam, in the unique solution for the combined footing system, both concepts should be implemented on the appropriate way, as well as their interaction conditions.

Because of complex nature of soils and peculiar aspects of their behavior and the complexity of boundary conditions involved in combined footing design it is necessary to use numerical methods for their analysis, because closed form mathematical solutions for such complex problems generally does not exist. In such cases the system of integral and differential equation should be solved numerically. In this case the numerical solution was obtained by finite difference and boundary element method. Presented numerical method implies that the solution of the problem always exists and therefore numerical procedure does not require iterative process.

Contact problem solution for flexible foundation beam.

It is worth to mention that unlike the traditional approach where the contact problem of foundation beam is treated as the planar one, here the same problem, as a part of the more broader one, it generally has to be treated as the space problem. Although formally identical to planar problem in symmetry conditions as in presented case, when considering the numerical solution of the soil same problem in space, one more spatial dimension has to be taken into account, when considering the interaction of surrounding footing base and footing structure.

Solution for the foundation beam (Fig. 2), which is here the flexible part of combined footing system, is based on the fourth order differential equation, according to the Euler-Bernoulli theory and has the following form

$$EI\frac{d^4w}{dx^4} = q(x) - p(x), \qquad (1)$$

where EI is flexural stiffness of the beam (constant in considered case), $w(x)$ function of the vertical displacement, $p(x)$ reactive stresses function on the contact area, the $q(x)$ is general external load function, including the beam self weight, E is elastic modulus of beam material, and I is the second moment of inertia of the beam cross-section taken about an axis perpendicular to the loading plane.

The theory is valid if assumed that the beam is relatively long and slender, material of the beam is isotropic, deformations of flexible beam remain small, there is no friction between beam and base on contact plane, beam is loaded in its symmetry plane, cross-section is constant along its axis and plane sections of the beam remain plane.

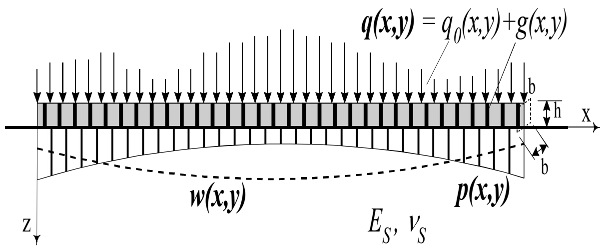


Figure 2 – Contact problem of the flexible foundation beam resting on elastic base

As the contact pressures p nor displacements w are not known, this leads to the staticly undetermined system so the additional assumptions are required to overcome this problem.

So for solving the spatial contact problem, it is supposed that vertical structure displacements and deformation of the base are equal, which leads to the boundary integral equation of the geometrical relationship between contact pressures p on the foundaton base and corresponding settlements w of the structure, wich generally has the folloving form

$$w(x,y) = \iint_A p(\xi,\eta)\omega(x,y,\xi,\eta)d\xi,d\eta, \quad (2)$$

where $w(x,y)$ is the function of unknown foundation displacements, A is the contact area of foundation resting on elastic base, $\omega(x, y, \xi, \eta)$ is a Green function depending on the adopted base model, $p(x, y)$ is the function of unknown contact pressures to be found [5],[6].

As it was already said for the foundaton beam, $w(x)$ and $p(x)$ are the functions of beam length only, providing the beam width is taken into account.

Differential equation is solved by fourfold integration whereat, it is necessary to specify geometrical (displacement w, inclination φ) or static (bending moment M, shear force T) boundary conditions to find specific solution.

Bending moments, and shear forces in arbitrary section plane of the foundation beam are defined as second and third order differential equations of the displacement function respectively

$$M(x) = -EI\frac{d^2w}{dx^2}, \quad (3)$$

$$T(x) = -EI\frac{d^3w}{dx^3}. \quad (4)$$

Equilibrium of the reaction forces and external load presented by equation

$$\iint_A p(\xi,\eta)d\xi,d\eta = \iint_A q(\xi,\eta)d\xi,d\eta, \quad (5)$$

is fulfilled on entire footing system domain A, but it is generally not fulfilled locally, on individual footing system part's subdomains A_n.

To be solved effectively, the problem has to be reduced to a discrete solution of the differential, integral and boundary condition equations in corresponding nodes.

The combined footing system (symmetrical in respect to the longitudinal axis) has been analyzed by the finite difference method on the simple orthogonal mesh of cells which dimensions in main directions X and Y are a and b respectively, and their corresponding nodes in the centre of cells, indicated by their discrete coordinates $j=1,...J$ and $i=1,...,I$ respectively (Fig. 3).

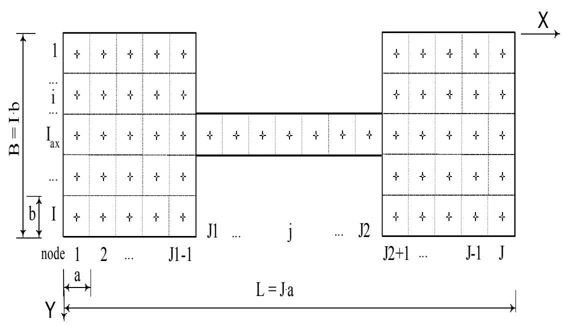


Figure 3 – Discretization scheme of the combined footing system

In this analysis the unique discretization scheme is introduced for the whole combined footing system in order to enable defining spatial relations in numerical expressions over a complete boundary of the problem. Discrete solution of differential equation of the foundation beam in the form of the system of finite difference equations can be presented in matrix form

$$\frac{EI}{a^4b}\cdot\mathbf{D}\times\mathbf{w} + \boldsymbol{\lambda}\times\mathbf{p} = \mathbf{f}, \quad (6)$$

where $\mathbf{D}$ is the matrix of differential operator of the foundation beam, $\mathbf{w}$ is the beam node displacement vector, $\boldsymbol{\lambda}$ is matrix dependent on discretization scheme (in case of mesh with central nodes it is unity matrix), $\mathbf{p}$ is the vector of reactive pressures, and $\mathbf{f}$ is the vector which consist of external normal load vector $\mathbf{q}$, and the load terms emerging from boundary condition equations.

Discrete analogue of spatial contact problem integral equation (2) is presented in discrete form of matrix equation:

$$\mathbf{w} = \mathbf{U} \times \mathbf{p}, \qquad (7)$$

where $\mathbf{U}$ is influence (deformability) matrix which present the vertical displacements of the contact surface at the arbitrary node (*i, j*), in the centre of corresponding cell on the boundary domain induced by the uniformly distributed unit load over another arbitrary cell (*k, l*) of the same boundary domain. Their elements in the case of elastic medium are obtained by the numerical integration of deformation resulting from the additional stresses distribution along the depth at node (*i, j*) induced by the unit load on the mesh cell (*k, l*), according to chosen contact model

Substituting $\mathbf{w}$ from equation (7) into equation (6) as we consider solving it in terms of contact pressure vector $\mathbf{p}$ to be the only unknown, the solution can be presented in the form

$$\mathbf{A} \times \mathbf{p} = \mathbf{f}, \qquad (8)$$

where

$$\mathbf{A} = \mathbf{D} \times \mathbf{U} + \boldsymbol{\lambda}, \qquad (9)$$

so the final solution of the contact problem of flexible foundation beam can be obtained in the form

$$\mathbf{p} = \mathbf{A}^{-1} \times \mathbf{f}. \qquad (10)$$

When final solution is found, the displacements can be obtained using (7) and the distributions of bending moments and shear forces along the foundation beam can be found by discrete finite difference approximations of equations (3) and (4) respectively in matrix form

$$\mathbf{M} = -EI \cdot \mathbf{D}_2 \times \mathbf{w}, \qquad (11)$$

$$\mathbf{T} = -EI \cdot \mathbf{D}_3 \times \mathbf{w}, \qquad (12)$$

where the $\mathbf{D_2}$ and $\mathbf{D_3}$ are finite difference analogue of second and third derivation operator respectively.

Unlike a free edge foundation beam, where the static boundary conditions would be homogenous (M=0, T=0) so specific solution would be rather simple to obtain, in the case of beam which is part of combined footing real bending moment ($M \neq 0$), and shear force ($T \neq 0$) have to be defined in boundary condition analysis in the sense of introduced load variables which include unknown reactive pressures too (Fig. 4).

Boundary bending moments on beam edge section planes (*n=1, 2*) can be defined as integral sum of differential force moments over the area *An* of adjacent rigid footing *n* in respect to corresponding section plane x_n

$$M(x_n) = \int_{An} q(\xi,\eta) - p(\xi,\eta)\xi d\xi, d\eta. \qquad (13)$$

Accordingly, the boundary shear forces on the same section planes can be defined as integral sum of differential forces over the area of corresponding rigid footing

$$T(x_n) = \int_{An} q(\xi,\eta) - p(\xi,\eta) d\xi, d\eta. \qquad (14)$$

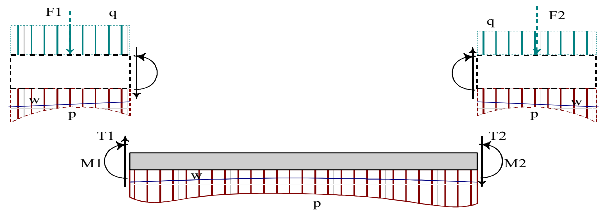


Figure 4 – Boundary conditions for flexible foundation beam interacting with rigid footing

which can be expressed in the discrete terms according to formerly introduced discretization scheme

$$M_n = F \cdot x_P + ab\sum_{k=1}^{I}\sum_{l=J1(n)}^{J2(n)}\left(q_{kl} \cdot x_{kl}\right) - ab\sum_{k=1}^{I}\sum_{l=J1(n)}^{J2(n)}\left(p_{kl} \cdot x_{kl}\right), \qquad (15)$$

$$T_n = F + ab\sum_{k=1}^{I}\sum_{l=J1(n)}^{J2(n)}(q)_{kl} - ab\sum_{k=1}^{I}\sum_{l=J1(n)}^{J2(n)} p_{kl}, \qquad (16)$$

where M_n and T_n are boundary bending moments and shear forces, x_P and x_{kl} the vector distances of the concentrated forces and mesh node forces respectively regarding to the edge section plane, *J1(n)* and *J2(n)* starting and final node points of the n - th footing length.

Introducing equations of the boundary conditions (13), (14) in their discrete form (15), (16) into the nodal difference equations

$$\frac{EI}{a^4 b} \cdot \left(w_{j-2} - 4w_{j-1} + 6w_j - 4w_{j+1} + w_{j+2}\right) = q_j - p_j, \qquad (17)$$

for two near-boundary beam nodes, which include deflection terms of the two extra - boundary nodes (Fig. 5) that can be called fictive displacements and are defined from before mentioned static boundary conditions

$$w_{J1-1} = -\frac{a^2}{EI}\left(M_1^T + M_1\right) + 2w_{J1} - w_{J1+1}, \qquad (18)$$

$$w_{J1-2} = \frac{2a^3}{EI}\left(T_1^T + T_1\right) + 2w_{J1-1} - 2w_{J1+1} + w_{J1+2}, \qquad (19)$$

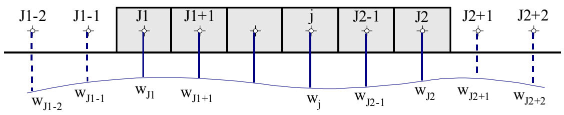


Figure 5 – Discretization scheme of the foundation beam including fictive displacements in extra-boundary nodes

so the numerical solution of differential equation for bending of the flexible foundation beam as a part of combined footing system is found an can be defined by the matrix equation

$$\left(\frac{EI}{a^4 b}\cdot \mathbf{D}^b \times \mathbf{U} + \boldsymbol{\lambda}^b + \mathbf{R}^b\right) \times \mathbf{p} = \mathbf{f}, \qquad (20)$$

where $\mathbf{D}^b$ is beam finite difference operator, $\mathbf{U}$ is the influence matrix, $\boldsymbol{\lambda}^b$ the discretization scheme dependent operator, $\mathbf{R}^b$ matrix of boundary condition terms which include unknown variables, and $\mathbf{f}$ the vector of external load and boundary condition load terms.

Solution of the contact problem for the rigid footing.

As the footings are considered absolutely rigid, there is no deformation of its shape, so contact surface remains plane exhibiting only kinematical movement which can be described by its inclination and displacement of the arbitrary point on the contact plane surface [1] and expressed in the form of integral equation

$$\int_A p(\xi,\eta)\omega(x,y,\xi,\eta)d\xi,d\eta = w_C + \varphi_x(x - X_C) + +\varphi_y(y - Y_C), \qquad (21)$$

where A is the contact area, w_C is vertical settlement of the footing centre, φ_x and φ_x are footing inclinations relative to axes X and Y respectively.

So, to solve the contact problem of rigid footing it is necessary to find a distribution of contact pressures and three unknown parameters of displacement and inclinations of rigid footing contact plane (w_C, φ_x, φ_y).

To find the solution, we need three more equilibrium equations:

$$\int_A p(\xi,\eta)d\xi,d\eta = F, \qquad (22)$$

$$\int_A p(\xi,\eta)d\xi,d\eta = F\cdot Xc + My, \qquad (23)$$

$$\int_A p(\xi,\eta)d\xi,d\eta = F\cdot Yc + Mx. \qquad (24)$$

The solution of the problem can be defined in discrete form

$$\sum_{k=1}^{I}\sum_{l=1}^{J} p_{k,l}\cdot u_{i,j,k,l} - \varphi_x(x - X_C) - \varphi_y(y - Y_C) + w_C = 0, \qquad (25)$$
$$(i,k = 1,\dots,I,\ j,l = 1,\dots,J).$$

As the spatial position of the contact plane can be determined by the displacement of arbitrary point n of its surface and its inclination, it is practical to define contact problem of rigid footing in terms of existing variables i.e. node point displacements only, so the solution (25) can be presented in a matrix form too, similar to finite difference operator.

The discrete solution of the problem can be defined as system of linear equations of kinematical conditions providing link between node displacements in respect to the longitudinal and transversal direction.

As the footing system is loaded centrically regarding to its longitudinal axis (X), all of the displacements in the normal direction to the longitudinal axis are equal i.e. $\varphi_y=0$ so the equation

$$w_{1,j} = w_{2,j} = \ \dots\dots\ = w_{I,j}, \qquad (26)$$

is valid for all node columns $j=1,\dots,J1-1$ for the first footing (Fig. 6), and also corresponding $j=J2+1,\dots,J$ on the second footing according to discretization scheme (Fig. 3), which leads to first set of kinematical conditions in the form

$$w_{1,j} - w_{2,j} = 0;\ w_{2,j} - w_{3,j} = 0;\ \dots\dots\ w_{I-1,j} - w_{I,j} = 0. \qquad (27)$$

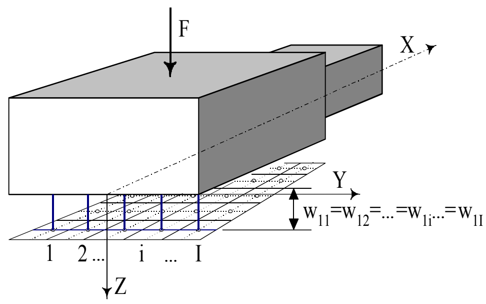


Figure 6 – Kinematical conditions for the rigid footing in the normal direction to the longitudinal axis

Combining this set of kinematical condition equations with the boundary integral equation (2) in their discrete matrix form (7) as in (25) we get the set of kinematical equations which can be expressed in matrix form

$$\mathbf{K}_1^f \times \mathbf{U} \times \mathbf{p}^f = \mathbf{0}. \qquad (28)$$

Kinematical conditions in the direction of longitudinal axis can be expressed as the constant inclination of footing slope φ_x which has to be defined only in central node row Iax, as it is the same for all other rows (Fig. 7)

$$\frac{w_{Iax,2}-w_{Iax,1}}{a}=\frac{w_{Iax,3}-w_{Iax,2}}{a}=....=$$
$$=\frac{w_{Iax,J1}-w_{Iax,J1-1}}{a}=\varphi_x, \quad (29)$$

which leads to second set of kinematical condition equations in the form

$$-w_{Iax,1}+2w_{Iax,2}-w_{Iax,3}=0,\ ...$$
$$-w_{Iax,j-1}+2w_{Iax,j}-w_{Iax,j+1}=0,\ ... \quad (30)$$
$$-w_{Iax,J1-2}+2w_{Iax,J1-1}-w_{Iax,j+1}=0.$$

Combining these equations with (7) we get the set of kinematical equations

$$\mathbf{K}_2^f \times \mathbf{U} \times \mathbf{p}^f = \mathbf{0}. \quad (31)$$

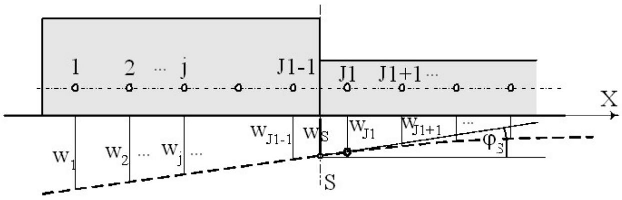


Figure 7 – Boundary conditions on the connection of the rigid footing with the foundation beam

Equations (29) and (30) are formally stated in the notation for the first footing but those for the second one are principally identical. By equations (30), (31) contact problem of rigid footing is defined.

Boundary plane conditions and solution for combined footing system.

Regarding to fixed static connection of the footing with the foundation beam, there are geometric boundary conditions of equal displacement and equal inclination which must be satisfied on their connection which can be defined as (Fig. 7)

$$w_S^f = w_S^b \neq 0, \quad (32)$$

$$\varphi_S^f = \varphi_S^b \neq 0, \quad (33)$$

where w_S is the displacement in the section plane. Index b and f indicate quantities regarding to beam and footing respectively.

As the edge section is not in the node in the case of mesh with central node points, the boundary displacement is approximated in terms of beam displacement variables

$$w_{Is,J1}^b = w_{J1}^f + \varphi_{J1}^f \cdot a. \quad (34)$$

The condition of equal inclination (33) is defined by combining central finite difference approximation of the footing slope inclination

$$\varphi_{J1}^f = \frac{dw}{dx} \approx \frac{\left(w_{J1}^f - w_{J1-1}^f\right)}{a}, \quad (35)$$

and boundary finite difference approximation of foundation beam slope inclination

$$\varphi_{Is,J1}^b = \frac{dw}{dx} \approx \frac{-11w_{Is,J1}^b + 18w_{Is,J1+1}^b - 9w_{Is,J1+2}^b + 2w_{Is,J1+3}^b}{6a}, \quad (36)$$

which yields

$$-6w_{Is,J1-1} + 17w_{Is,J1} - 18w_{Is,J1+1} + 9w_{Is,J1+2} - 2w_{Is,J1+3} = 0, \quad (37)$$

while the condition of the equal displacement (32) is already fulfilled by the equation (33).

Combining equation (37) with (7) we get two kinematical equations which can be noted as

$$\mathbf{K}_3^f \times \mathbf{U} \times \mathbf{p}^f = \mathbf{0}. \quad (38)$$

Equations (28), (33) and (38) define complete solution for rigid footing parts of combined footing system

$$\mathbf{K}^f \times \mathbf{U} \times \mathbf{p}^f = \mathbf{0}, \quad (39)$$

where $\mathbf{K}^f$ is complete kinematical operator matrix of the rigid footing parts of combined footing system and $\mathbf{p}^f$ is the vector of contact stresses on rigid footings contact plane.

Finally, the unique solution for the strip footing system is defined by combining the solutions for his flexible (20) and rigid (39) parts and can be described by the matrix equation

$$\left[\left(\frac{EI}{a^4 b} \cdot \mathbf{D}^b + \mathbf{K}^f\right) \times \mathbf{U} + \boldsymbol{\lambda}^b + \mathbf{R}^b\right] \times \mathbf{p} = \mathbf{f}. \quad (40)$$

Schematic graphical presentation of the sparse matrix system components is shown on Fig. 8.

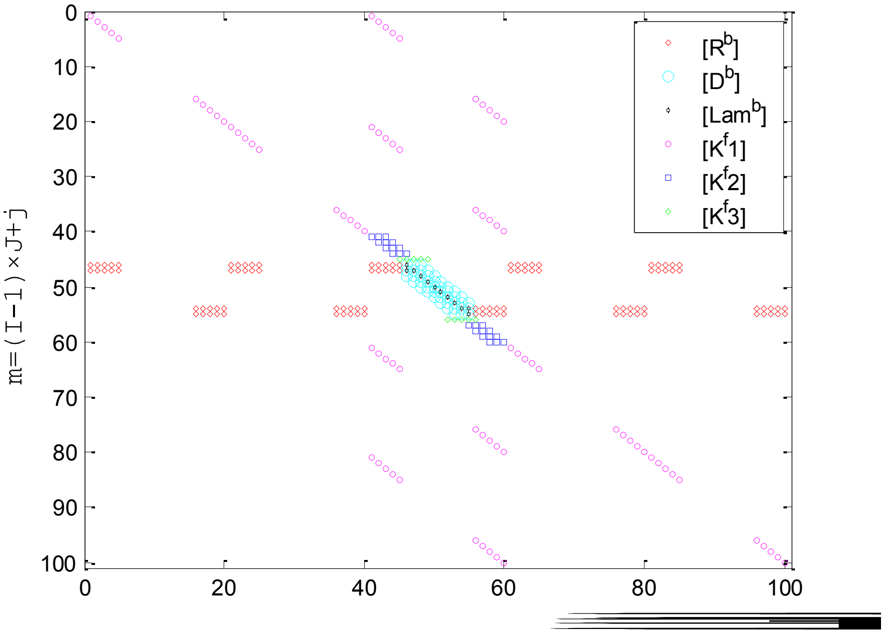


Figure 8 – Sparse matrix system components of finite difference solution for strip footing system analysis

Soil model.

Solution of the contact problem on elastic homogenous isotropic half-space is based on well known Bousinesq's solution for additional stresses due to concentrated load on its surface [18].

The relationship (2) between uniformly distributed contact pressure p on the foundation base and corresponding settlement w on the corner point of the rectangular area of dimensions (a,b) resting on the elastic half-space is obtained by double integration of differential deformation caused by additional stresses over the area according to Steinbrenner's solution [9].

$$w=\frac{1-\nu_0^2}{\pi\cdot E_o}\cdot p\cdot\left[a\cdot\ln\frac{a}{\sqrt{a^2+b^2}-b}+b\cdot\ln\frac{\sqrt{a^2+b^2}-a}{b}\right]. \quad (41)$$

Settlement of any node (i, j) is evaluated by numerical integration of partial settlements caused by any single uniformly loaded rectangular mesh element area (k, l) of foundation base boundary by superposition of solution (41) for all divided rectangular element area parts with common corner in it's characteristic point.

As an alternative to the elastic homogenous isotropic half - space model, the quasi-linear elastic layered soil model is also adopted to enable using of standard widely used and experimentally determined soil properties as the confined compressibility modulus based on Edometer test data. Total settlement of the compressible soil layer is defined by

$$w=\int_0^H\frac{\sigma_z\cdot dz}{M_{V(z)}}, \quad (42)$$

where H is the soil layer thickness, $M_{V(z)}$ is confined compressibility soil modulus on corresponding depth z and σ_z is additional stress function.

For actual calculations this integral equation (2) solution is evaluated numerically in discretized form over the depth of soil profile which is divided in many (N) thin sub-layers of thickness $\Delta z(n)$ approximated by constant soil compression modulus $M_{V(n)}$ and stress $\sigma_{z(n)}$ which is calculated in the centre of sub-layer depth

$$w=\sum_1^N\frac{\sigma_{z(n)}\cdot\Delta z(n)}{M_{V(n)}}, \quad n=1\ldots N. \quad (43)$$

Additional stress function is defined according to the Steinbrenner's solution for stress on depth z below the edge of the rectangular area of dimensions (a, b)

$$\sigma_z=\frac{q}{2\pi}\left[\frac{a\cdot b\cdot z\cdot\left(a^2+b^2+2z^2\right)}{\left(a^2+z^2\right)\left(b^2+z^2\right)\sqrt{a^2+b^2+z^2}}+arctg\frac{a\cdot b}{z\cdot\sqrt{a^2+b^2+z^2}}\right], \quad (44)$$

which is evaluated on any point of rectangular mesh sub domain using the superposition principle as described before.

Numerical examples.

The numerical algorithm based on above described procedure for combined footing system analysis was written and implemented in Matlab computing environment. An elastic linearly deformed half-space and quasi-linear elastic layered model of horizontally layered soil are implemented in the algorithm based on finite difference method.

Two simple examples of symmetrically loaded combined footing system are analyzed to demonstrate the adopted procedure.

The footing system domain was described by uniform rectangular mesh of 20×5 cells which dimensions in main directions *X* and *Y* are *a=b=0.5m.*

Basic footing system of overall length of 10.0 m consists of two rigid footings of dimensions (w×l×h) 2.5×2.5×0.6 m, connected with the foundation beam of dimensions 0.5×5.0×0.6 m. Footing system model mesh consist of 60 cells with node points in their center, so the total number of unknowns in the problem is 60.

Graphical presentation of the combined footing system model with its dimensions over the corresponding contact area mesh is given on Fig. 9.

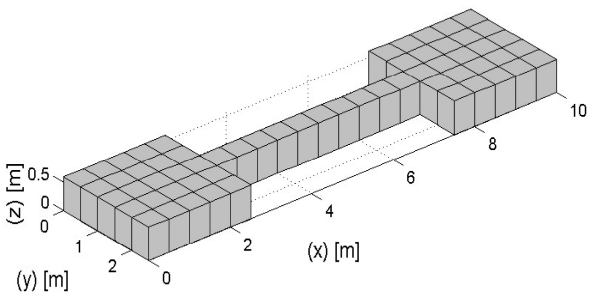


Figure 9 – The model of the combined footing system

Footing system is loaded by external concentrated forces P1=P2=600 kN in the centre of each footing and self weigh 225 kN, so the total load sum is 1425 kN. Concrete Young modulus is E=3×10^4 MPa and Poisson ratio ν=0.17.

For a given load combined footing load contact pressure distribution and the node displacements are calculated. Bending moments and shear forces in node points and characteristic sections are calculated by integration of node forces.

The numerical calculation data of contact pressure distribution and node displacements are given for a quarter of footing system area, and bending moments and shear forces in longitudinal axis for half footing system length, due to symmetry of load distribution.

Calculation data for combined footing system on elastic half-space.

For this calculation the elastic half-space soil model was characterized by Young modulus E=10 MPa and Poisson ratio ν=0.3.

The numerical calculation data are presented in Table 1. Calculation results for contact pressure distribution and node displacements are shown graphically over footing system domain on the figures Fig. 10 and Fig. 11. Bending moments and shear forces in footing system longitudinal axis are shown on the Fig. 12.

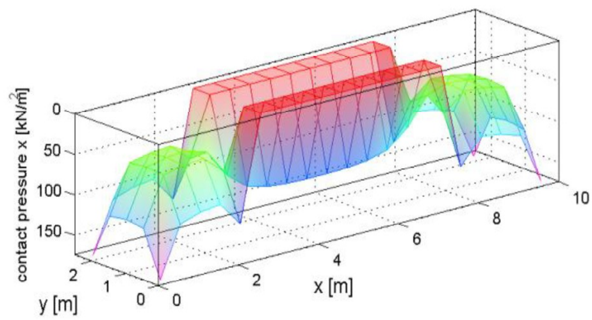


Figure 10 – Combined footing contact pressure distribution for the elastic half-space soil model

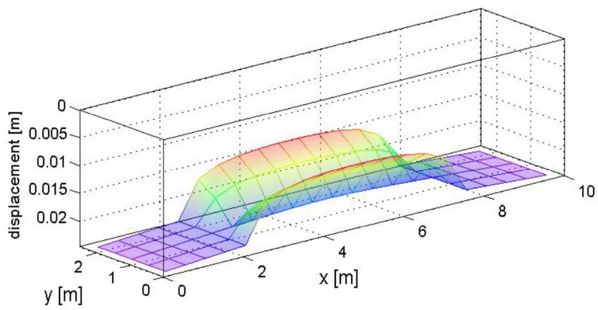


Figure 11 – Combined footing node displacements for the elastic half-space soil model

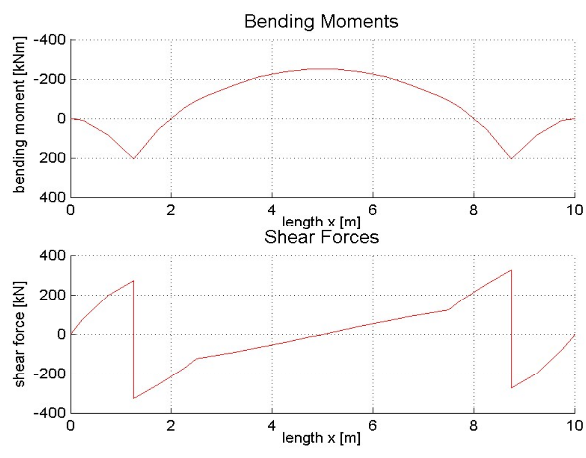


Figure 12 – Combined footing bending moments and shear forces distribution for the elastic half-space soil model

Table 1 – Numerical calculation data for combined footing on the elastic half-space

Node (i, j)	cont. pressure p [kN/m^2]	displacement w [m]
1,1	174.375	0.024597
1,2	111.9333	0.024346
1,3	103.4954	0.024096
1,4	100.1282	0.023846
1,5	130.6441	0.023596
2,1	118.3422	0.024597
2,2	58.8469	0.024346
2,3	53.4642	0.024096
2,4	51.1634	0.023846
2,5	73.2836	0.023596
3,1	115.0437	0.024597
3,2	56.7569	0.024346
3,3	51.1526	0.024096
3,4	49.1034	0.023846
3,5	57.4027	0.023596
3,6	92.7779	0.023346
3,7	112.4768	0.023093
3,8	119.3815	0.02286
3,9	121.8903	0.022684
3,10	122.6614	0.02259

length x [m]	Bending moment M [kNm]	Shear force N [kN]
0	0	0
0.25	9.7731	78.1848
0.75	83.2366	196.7842
1.25	205.8489	273.4578
1.25	205.8489	-326.5422
1.75	55.9658	-255.6974
2.25	-52.8154	-172.3293
2.5	-89.7999	-123.547
2.75	-119.4714	-113.8247
3.25	-171.2148	-91.9179
3.75	-210.9736	-66.6856
4.25	-237.7533	-40.2766
4.75	-251.199	-13.4577
5	-252.8812	-6.35E-08
Total load	1425	kN
Beam load Sum	37.500	kN
Beam contact pressures Sum	284.594	kN
Beam load vs. contact pressures difference	247.094	kN
Relative load transfer rigid foot. to beam	17.809	%

Calculation data for the combined footing system on the layered quasi-linear elastic soil model.

Horizontally layered quasi-linear elastic soil model consists of 4 layers with parameters specified in the Table 2.

Table 2 – Layered quasi-linear elastic soil model properties

Layer	Layer thickness [m]	Compressibility modulus [kN/m^2]
1	2	10000
2	2	15000
3	3	20000
4	3	30000

The numerical calculation data are presented in table 3. Calculation results for contact pressure distribution and node displacements are shown graphically over footing system domain on the figures Fig 13 and Fig 14. Bending moments and shear forces in longitudinal footing system axis are shown on the Fig 15.

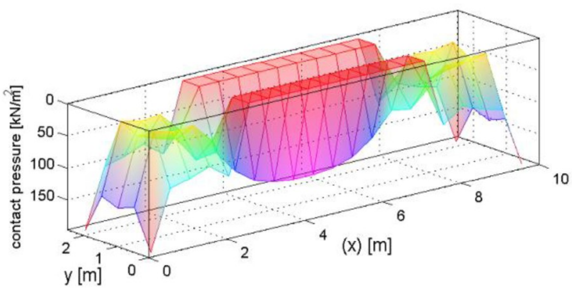


Figure 13 – Combined footing contact pressure distribution for layered quasi-linear elastic soil model

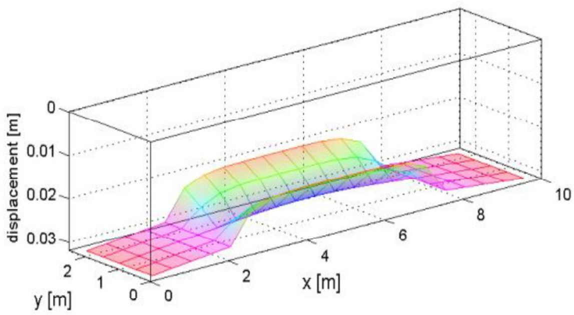


Figure 14 – Combined footing node displacements for layered quasi-linear elastic soil model

Table 3 – Numerical calculation data for combined footing on the layered quasi-linear elastic soil model

Node (i, j)	cont. pressure p [kN/m^2]	displacement w [m]
1,1	197.139	0.03188
1,2	104.493	0.03141
1,3	96.624	0.03094
1,4	84.640	0.03048
1,5	107.079	0.03001
2,1	135.362	0.03188
2,2	25.294	0.03141
2,3	32.381	0.03094
2,4	29.649	0.03048
2,5	60.898	0.03001
3,1	135.362	0.03188
3,2	145.777	0.03188
3,3	40.390	0.03141
3,4	51.352	0.03094
3,5	59.495	0.03048
3,6	99.214	0.03001
3,7	94.701	0.02954
3,8	130.879	0.02908
3,9	152.901	0.02868
3,10	162.377	0.02838

length x [m]	Bending moment M [kNm]	Shear force N [kN]
0	0.000	0.000
0.25	11.497	91.972
0.75	95.488	212.065
1.25	219.389	269.481
1.25	219.389	-330.519
1.75	64.783	-274.590
2.25	-56.897	-202.935
2.5	-102.003	-157.914
2.75	-140.236	-147.951
3.25	-208.665	-123.504
3.75	-262.830	-91.781
4.25	-299.954	-56.122
4.75	-318.750	-18.850
5	-321.106	-1.035E-08
Total load	1425	kN
Beam load Sum	37.500	kN
Beam contact pressures Sum	353.327	kN
Beam load vs. contact pressures difference	315.827	kN
Relative load transfer rigid foot. to beam	22.762	%

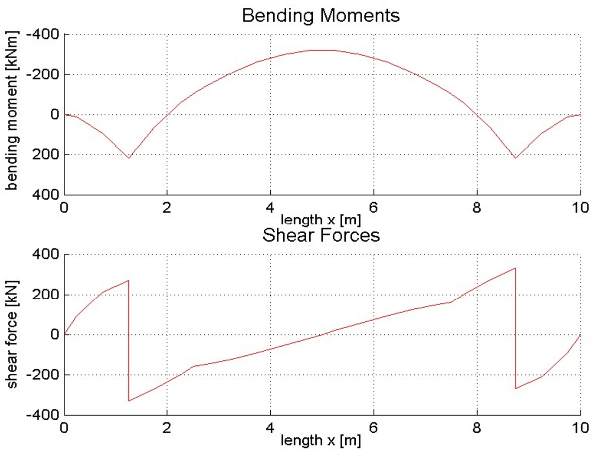


Figure 15 – Combined footing bending moments and shear forces distribution for layered quasi-linear elastic soil model

To evaluate influence of footing system rigidity on system parts mutual interaction effects, the parametric analysis was conducted varying the beam height. Numerical calculation data of basic footing design parameters for different beam heights are presented in table 4. Graphical presentation of parametric analysis results are shown on the figures Fig. 16 and Fig. 17.

The contact pressure distribution differences between two soil models are generally moderate and are the most visible on the flexible foundation beam where maximum differences for presented examples are of the order of 25 %.

The parametric analysis of footing system conducted by varying the beam height in the larger range of beam thickness, the load transfer from the rigid footings to the foundation beam has been changed less than 10%, which shows that the influence of foundation beam rigidity has does not have crucial influence on the foundation interaction effects.

As can be seen from analysis data the chosen soil model has more than twice as much influence on obtained foundation interaction effects than foundation beam stiffness which in the case of used soil models amounts up to 20 %.

Differential settlements are more dependent on foundation beam stiffness and maximum differential node settlements over the contact plane for analyzed examples varies in the range of about 10% to 20% of total displacement.

Bending moments and shear forces differences distribution also seem to depend more on used soil model than on the foundation beam rigidity

Table 4 – Comparison of calculation data for different beam height

Beam height [m]	Max. beam centre bend. moment [kNm]	Max. beam edge shear force [kN]	Max. differential node displ. [mm, (%)]	Load transfer to beam [kN, (%)]
[1]0.3	-201.9	312.1	5.15, (20.45)	225.4, (16.25)
[1]0.4	-221.5	316.6	3.89, (15.54)	234.6, (16.91)
[1]0.5	-239.5	322.9	2.83, (11.43)	242.1, (17.45)
[1]0.6	-252.9	326.5	2.01, (8.159)	247.1, (17.81)
[2]0.6	-321.1	330.5	3.50, (12.33)	315.8, (22.76)
[1] elastic linear half-space model; [2] quasi-linear elastic layered soil medium				

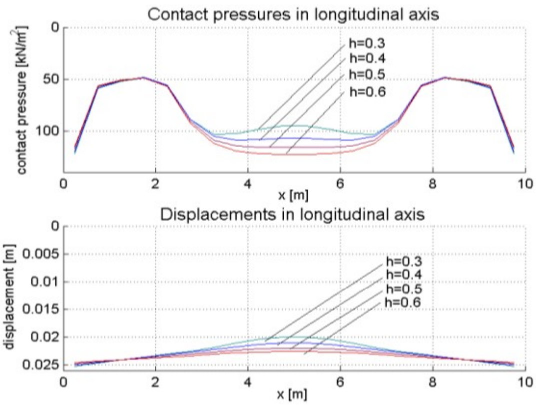


Figure 16 – Comparison of contact pressures and displacements for combined footing on elastic half-space depending on different beam height

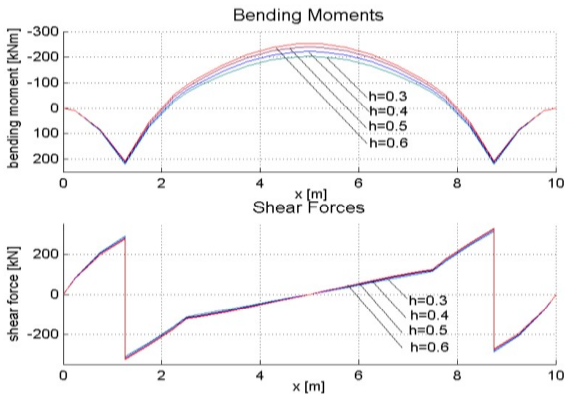


Figure 17 – Comparison of bending moments and shear forces for combined footing on elastic half-space depending on different beam height

Conclusions

In this work the procedure for the combined footing analysis using the finite difference method is presented.

Analysis results show that more realistic contact pressure distribution can be determined by taking into account mutual interaction of the rigid and flexible parts of combined footing system.

From presented results it is shown that interaction can cause significant amount of load to be redistributed from rigid footings to the foundation beam so the bending moments and shear forces distributions are affected by the interaction too.

It is quite interesting that the foundation beam rigidity change in the rather large scale does not have so significant influence on the interaction as it could be supposed. From conducted parametric analysis can be concluded that the analysis results more depend on implemented soil model.

Obtained results indicate that mutual interaction of the rigid and flexible parts of combined footing should be taken into consideration when calculating combined foundation systems in shallow foundation design to account for mutual interaction effects of the footing system parts to determine bending moments and shear forces accurately. In the conditions of inaccurately evaluated bending moments and shear forces construction can be exposed to the excess tensional and shear stresses mo, which can lead to the crack occurrence.

Although in the presented analysis relatively simple soil models are used it can be further improved by implementing more refined soil models, to enable taking into account more complex soil properties.

The conducted analysis demonstrate that finite difference method can be sufficiently well be adapted for meeting the boundary conditions of combined foundation system. It is also shown that developed algorithm is applicable for analysis of the combined shallow foundation system in order to determine contact stresses and displacements distributions which is necessary for foundation design. Accordingly, the other parameters for foundation design as bending moments and shear forces can be determined more accurately.

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КОНТАКТНАЯ ЗАДАЧА КОМБИНИРОВАННОЙ ОПОРЫ, КОТОРАЯ ОСНОВЫАВЕТСЯ НА УПРУГОМ ПОЛУПРОСТРАНСТВЕ

Была проанализирована контактная задача комбинированной опоры, которая основывается на упругом полупространстве, или стратифицированном упругом основании. В отличие от упрощенных общих предположений и методов проектирования такого основания, здесь анализ был проведен на двух жестких опорах, соединенных упругой гибкой фундаментальной балкой с учетом различных характеристик отдельных частей фундамента (жесткие против гибких) и условия непрерывности на их соединениях, с акцентом на их взаимовлияния, связанные с основным распределением нагрузки почвы. Численный анализ был выполнен комбинированием конечных разностей и методом граничных элементов на одинаковой четырехсторонней сетке.

Ключевые слова: *комбинированный фундамент, взаимодействие, упругое полупространство, упругая многослойная среда, метод конечных элементов, метод граничных элементов, согласование.*

Томіслав Копрек, Божо Солдо, д.т.н., **Олексій Аніскін**

КОНТАКТНА ЗАДАЧА КОМБІНОВАНОЇ ОПОРИ, ЩО ҐРУНТУЄТЬСЯ НА ПРУЖНОМУ ПІВПРОСТОРІ

Була проаналізована контактна задача комбінованої опори, яка ґрунтується на пружному півпросторі, або стратифікованій пружній основі. На відміну від спрощених загальних припущень і методів проектування такої основи, тут аналіз був проведений на двох жорстких опорах, сполучених пружною гнучкою фундаментальною балкою з урахуванням різних характеристик окремих частин

фундаменту (жорсткі проти гнучких) і умови безперервності на їх з'єднаннях, з акцентом на їх взаємовплив, пов'язані з основним розподілом навантаження ґрунту. Чисельний аналіз був виконаний комбінуванням кінцевих різниць і методом граничних елементів на однаковій чотиристоронній сітці.

Ключові слова: *комбінований фундамент, взаємодія, пружний півпростір, пружна багатошарова середа, метод кінцевих елементів, метод граничних елементів, узгодження.*

УДК 519.878.5

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МЕТОД ГРАНИЧНЫХ ЭЛЕМЕНТОВ В ЗАДАЧАХ УСТОЙЧИВОСТИ ПЛОСКОЙ ФОРМЫ ИЗГИБА БАЛОК ПРЯМОУГОЛЬНОГО СЕЧЕНИЯ.

Представлен алгоритм решения задач устойчивости плоской формы изгиба балок прямоугольного сечения (тонкой полосы) с помощью численно-аналитического варианта метода граничных элементов. Целью работы является построение новых решений дифференциальных уравнений задач устойчивости. Балки с сечениями в виде узкой полосы имеют более высокую прочность и жесткость, однако, при поперечной нагрузке, возникает опасность потери устойчивости плоской формы изгиба. В этом случае балка дополнительно изгибается в другой плоскости и закручивается. Возникает изгибно-крутильная форма потери устойчивости, при которой появляются большие перемещения и может наступить разрушение конструкции. Теория решения подобных задач нуждается в развитии, т.к. существующие результаты весьма сложно распространить на неразрезные балки и рамы. Метод граничных элементов позволяет существенно упростить процесс решения, повысить точность и достоверность результатов и распространить полученные решения на более сложные конструкции, чем просто балки. Расчеты критических сил выполнены в среде MATLAB.

Ключевые слова: *метод граничных элементов, устойчивость плоской формы изгиба, балки прямоугольного сечения, MATLAB.*

Балки с сечением в виде узкой полосы имеют более высокую прочность и жесткость. В этой связи они имеют большое применение в различных балочных и рамных конструкциях машиностроения, строительства, мостов и т.д. Однако, при большом отношении высоты к ширине сечения, возникает реальная опасность потери устойчивости плоской формы изгиба такой балки. Она дополнительно получает прогибы в другой плоскости и углы закручивания. Если перемещения оказываются слишком большими, конструкция разрушается. Поэтому весьма важно иметь надежную, достоверную и достаточно простую теорию решения таких задач устойчивости.

Первые решения задач устойчивости плоской формы изгиба балок с сечением в виде узкой полосы были получены еще в 19 веке [1]. К настоящему времени решено достаточно много задач этого типа [2, 3]. При решении задач устойчивости балок под действием поперечной нагрузки дифференциальное уравнение и его решение записывалось для угла закручивания. В решении использовались функции, представляющие собой бесконечные знакочередующиеся степенные ряды. В этом случае точность результата зависит от числа членов ряда, что не всегда удобно. Кроме того, имеющиеся решения весьма сложно использовать для решения задач устойчивости неразрезных балок и рам, поскольку они неполные (не включают другие параметры). Если же применить к этим задачам алгоритм численно-аналитического метода граничных элементов (МГЭ) [4], то появится возможность существенно упростить процедуру решения, повысить точность и достоверность результатов и применять новые решения дифференциальных уравнений в более сложных конструкциях, чем простые балки.

Целью работы является построение новых решений дифференциальных уравнений задач устойчивости плоской формы изгиба балок с сечением в виде узкой полосы и применение этих решений в конкретных задачах.

Дифференциальное уравнение устойчивости двутавровой балки вывел С. П. Тимошенко [1].