ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2015, **7** (2), 247–253
doi:10.15330/cmp.7.2.247-253

http://www.journals.pu.if.ua/index.php/cmp
Карпатські матем. публ. 2015, Т.7, №2, С.247–253

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ON PROPERTIES OF THE SOLUTIONS OF THE WEBER EQUATION

Growth, convexity and the l-index boundedness of the functions $\alpha(z)$ and $\beta(z)$, such that $\alpha(z^4)$ and $z\beta(z^4)$ are linear independent solutions of the Weber equation $w'' - (\frac{z^2}{4} - \nu - \frac{1}{2})w = 0$ with $\nu = -\frac{1}{2}$ are investigated.

Key words and phrases: entire function, l-index boundedness, convex function, growth, Weber equation.

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INTRODUCTION

Let

$$f(z) = \sum_{n=0}^{\infty} f_n z^n \tag{1}$$

be an entire function, l — a positive continuous on $[0, +\infty)$ function. Function f is said to be of bounded l-index [3], if there exists $N \in \mathbb{Z}_+$ such that for all $n \in \mathbb{Z}_+$ and $z \in \mathbb{C}$

$$\frac{|f^{(n)}(z)|}{n! l^n(|z|)} \le \max\left\{\frac{|f^{(k)}(z)|}{k! l^k(|z|)} : 0 \le k \le N\right\}. \tag{2}$$

The least such integer N is called l-index and is denoted by $N(f, l)$. If $G \subset \mathbb{C}$ and there exists $N \in \mathbb{Z}_+$ such that inequality (2) holds for all $n \in \mathbb{Z}_+$ and $z \in G$, analytic in G function f is said to be of bounded l-index on (or in) G, and l-index is denoted by $N(f, l; G)$. Theorem 2.2 [3, p.33] implies that *if f is an entire function, G is a bounded domain and l — a positive continuous function, then f is of bounded l-index in G.*

An analytic univalent in $\mathbb{D} = \{z : |z| < 1\}$ function (1) is said to be convex if $f(\mathbb{D})$ is a convex domain. Condition $\operatorname{Re}\{1 + zf''(z)/f'(z)\} > 0\,(z \in \mathbb{D})$ is necessary and sufficient [1] for a convexity of f. Every convex function is univalent in $\mathbb{D}$, and therefore $f_1 \neq 0$.

Differential equation

$$w'' - \left(\frac{z^2}{4} - \nu - \frac{1}{2}\right) w = 0 \tag{3}$$

is said to be the Weber equation. Properties of the solutions of the Weber equation if $\nu \neq -\dfrac{1}{2}$ are investigated in [5] and the following theorem is proved.

УДК 517.5
2010 *Mathematics Subject Classification:* 34M05, 30D15.

Theorem ([5]). *If $\nu \neq -\dfrac{1}{2}$ the general solution of the equation (3) is of the form $w(z) = C_1\varphi(z^2) + C_2 z\psi(z^2)$, and the functions $\varphi(z)$ and $\psi(z)$ have the following properties:*

1) *$N(\varphi, l) \leq 1$ with $l(|z|) \equiv \dfrac{28}{9}|2\nu+1| + \dfrac{18}{5}$ and $N(\psi, l) \leq 1$ with $l(|z|) \equiv \dfrac{11}{10}(|2\nu+1|+2)$;*

2) *if $(762 - \sqrt{388564})/343 \leq |2\nu+1| \leq (762+\sqrt{388564})/343$, then $\varphi(z)$ is convex in $\mathbb{D}$, and if $(2350 - \sqrt{3590164})/639 \leq |2\nu+1| \leq (2350+\sqrt{3590164})/639$ then $\psi(z)$ is convex in $\mathbb{D}$;*

3) *if $(1623 - \sqrt{2430289})/364 \leq |2\nu+1| \leq (1623+\sqrt{2430289})/364$, then $\varphi(z)$ is close-to-convex in $\mathbb{D}$, and if $(4915 - \sqrt{22088809})/684 \leq |2\nu+1| \leq (4915+\sqrt{22088809})/684$, then $\psi(z)$ is close-to-convex in $\mathbb{D}$;*

4) *for $\nu \in \mathbb{R}$ if $(-7-\sqrt{34})/2 \leq \nu \leq (-7+\sqrt{34})/2$ the function $\varphi(z)$ is close-to-convex in $\mathbb{D}$, and if $(-11-\sqrt{94})/2 \leq \nu \leq (-11+\sqrt{94})/2$ the function $\psi(z)$ is close-to-convex in $\mathbb{D}$;*

5) *$\ln M_\varphi(r) = (1+o(1))\dfrac{r}{4}$ and $\ln M_\psi(r) = (1+o(1))\dfrac{r}{4}$ as $r \to \infty$, where*
$$M_f(r) = \max\{|f(z)| : |z| = r\}.$$

In this article we consider the case $\nu = -\frac{1}{2}$. Then from (3) we have
$$w'' - \frac{z^2}{4}w = 0. \tag{4}$$
Let us find the solution of the equation (4) in the form (1). Since
$$\sum_{n=0}^{\infty}(n+1)(n+2)f_{n+2}z^n - \frac{1}{4}\sum_{n=2}^{\infty} f_{n-2}z^n = 0,$$
so $2f_2 = 0$, $6f_3 = 0$ and $4(n+2)(n+1)f_{n+2} = f_{n-2}$ if $n \geq 2$. We can see that for all $n \in \mathbb{N}$ $f_{4n-2} = f_{4n-1} = 0$, and f_{n+4} depends on f_n. Therefore the solution of the equation (4) is of the form
$$w(z) = C_1\alpha(z^4) + C_2 z\beta(z^4).$$
Let $w(z) = \alpha(z^4)$. Then $w'(z) = 4z^3\alpha'(z^4)$, $w''(z) = 12z^2\alpha'(z^4) + 16z^6\alpha''(z^4)$, and, therefore, the equation (4) in this case is of the form $16z^6\alpha''(z^4) + 12z^2\alpha'(z^4) - \frac{z^2}{4}\alpha(z^4) = 0$. After elementary transformations and replacement z^4 on z we will get
$$64z\alpha''(z) + 48\alpha'(z) - \alpha(z) = 0. \tag{5}$$
If we suppose that $w(z) = z\beta(z^4)$, then, like before, we will get
$$64z\beta''(z) + 80\beta'(z) - \beta(z) = 0. \tag{6}$$
We will find a recurrent formula for the coefficients of the function $\alpha(z) = \sum\limits_{n=0}^{\infty}\alpha_n z^n$, which is the solution of the equation (5). Since
$$64\sum_{n=1}^{\infty}\alpha_{n+1}(n+1)nz^n + 48\sum_{n=0}^{\infty}\alpha_{n+1}(n+1)z^n - \sum_{n=0}^{\infty}\alpha_n z^n = 0,$$

we equate coefficients at the same powers of the variable z and get $48\alpha_1 - \alpha_0 = 0$ and $(64n(n+1) + 48(n+1))\alpha_{n+1} - \alpha_n = 0$ if $n \geq 1$. Note, if $\alpha_0 = 0$ then $\alpha(z) \equiv 0$. Thus we put $\alpha_0 = 1$. Then

$$\alpha_n = \frac{\alpha_{n-1}}{16n(4n-1)}, \quad n \geq 1. \tag{7}$$

For the coefficients of the function $\beta(z) = \sum\limits_{n=0}^{\infty} \beta_n z^n$ which is the solution of the equation (6), we have $80\beta_1 - \beta_0 = 0$ and $16(n+1)(4n+5)\beta_{n+1} - \beta_n = 0$ if $n \geq 1$. If we put $\beta_0 = 1$ then

$$\beta_n = \frac{\beta_{n-1}}{16n(4n+1)}, \quad n \geq 1. \tag{8}$$

1 l-INDEX BOUNDEDNESS

Now we consider l-index boundedness of the functions $\alpha(z)$ and $\beta(z)$. For this purpose we use the following lemma.

Lemma 1 ([4]). *If a function* (1) *is an analytic in the closed disc* $\overline{\mathbb{D}}_R = \{z : |z| \leq R\}$, $f_0 = 1$, *and*

$$\sum_{n=1}^{\infty} |f_n| R^n \leq a(R) < 1, \tag{9}$$

then $N(f, l; \mathbb{D}_R) \leq 1$ *with* $l(|z|) = \dfrac{1 + a(R)}{(1 - a(R))(R - |z|)}$.

If $z \in \mathbb{D}_{\xi R}$, $0 < \xi < 1$, then $R - |z| \geq (1 - \xi)R$ and Lemma 1 implies $N(f, l; \mathbb{D}_{\xi R}) \leq 1$ with $l(|z|) \equiv \dfrac{1 + a(R)}{(1 - \xi)R(1 - a(R))}$, because if $N(f, l_*; G) \leq N$ and $l_*(r) \leq l^*(r)$, it is easy to prove [3, p.23], that $N(f, l^*; G) \leq N$. Therefore the next lemma is true.

Lemma 2. *If an entire function* (1) *satisfies* (9) *and* $f_0 = 1$, *then for every* $\xi \in (0, 1)$ *and* $R \in (0, +\infty)$ *the inequality* $N(f, l; \mathbb{D}_{\xi R}) \leq 1$ *holds with* $l(|z|) \equiv \dfrac{1 + a(R)}{(1 - \xi)R(1 - a(R))}$.

Using (7) we have

$$\sum_{k=1}^{\infty} |\alpha_k| R^k = \frac{R}{48} + \sum_{k=2}^{\infty} |\alpha_k| R^k = \frac{R}{48} + \sum_{k=2}^{\infty} \frac{|\alpha_{k-1}| R^k}{16k(4k-1)} = \frac{R}{48} + \sum_{k=1}^{\infty} \frac{R}{16(k+1)(4k+3)} |\alpha_k| R^k.$$

That is

$$\sum_{k=1}^{\infty} \left(1 - \frac{R}{16(k+1)(4k+3)}\right) |\alpha_k| R^k = \frac{R}{48}.$$

Since $\dfrac{R}{16(k+1)(4k+3)} \leq \dfrac{R}{224}$, if $R < 224$, then above equality implies

$$\sum_{k=1}^{\infty} |\alpha_k| R^k \leq \frac{R/48}{1 - (R/224)} = \frac{14R}{672 - 3R} = a(R).$$

Therefore, to use Lemma 2 it is necessary $\dfrac{R}{48} + \dfrac{R}{224} < 1$. That is $R < \dfrac{672}{17}$. For such R by Lemma 2 we have $N(\alpha, l; \mathbb{D}_{\xi R}) \leq 1$ with $l(|z|) \equiv \dfrac{672 + 11R}{(1 - \xi)R(672 - 17R)}$.

Now we consider l-index boundedness of the function $\alpha(z)$ in $\mathbb{C}\setminus\mathbb{D}_{\xi R}$. For this purpose we use the fact that $\alpha(z)$ satisfies differential equation (5), and therefore we have $\alpha''(z) = -\frac{3}{4z}\alpha'(z) + \frac{1}{64z}\alpha(z)$. If $|z| \geq \xi R$, $l = 1/(\xi R)$ and $R < 672/17$, then we obtain

$$\frac{|\alpha''(z)|}{2!l^2} \leq \frac{3}{8}\frac{|\alpha'(z)|}{1!l} + \frac{\xi R}{128}|\alpha(z)| \leq \max\left\{\frac{|\alpha'(z)|}{1!l},\ |\alpha(z)|\right\}. \tag{10}$$

Let us differentiate the equation (5) n times. Then we obtain

$$64z\alpha^{(n+2)}(z) + (64n+48)\alpha^{(n+1)}(z) - \alpha^{(n)}(z) = 0.$$

Thus, if $|z| \geq \xi R$, $l = 1/(\xi R)$ and $R < 672/17$, then for all $n \geq 1$ we get

$$\begin{aligned}\frac{|\alpha^{(n+2)}(z)|}{(n+2)!l^{n+2}} &\leq \frac{64n+48}{64(n+2)|z|l}\frac{|\alpha^{(n+1)}(z)|}{(n+1)!l^{n+1}} + \frac{1}{64(n+2)(n+1)|z|l^2}\frac{|\alpha^{(n)}(z)|}{n!l^n}\\ &\leq \frac{64n+48}{64(n+2)}\frac{|\alpha^{(n+1)}(z)|}{(n+1)!l^{n+1}} + \frac{\xi R/(n+1)}{64(n+2)}\frac{|\alpha^{(n)}(z)|}{n!l^n} \leq \max\left\{\frac{|\alpha^{(n+1)}(z)|}{(n+1)!l^{n+1}},\ \frac{|\alpha^{(n)}(z)|}{n!l^n}\right\}.\end{aligned} \tag{11}$$

Inequalities (10) and (11) imply for all $n \in \mathbb{Z}_+$ and $z \in \mathbb{C}\setminus\mathbb{D}_{\xi R}$

$$\frac{|\alpha^{(n)}(z)|}{n!l^n} \leq \max\left\{\frac{|\alpha'(z)|}{1!l},\ |\alpha(z)|\right\},$$

that is, $N(\alpha, l; \mathbb{C}\setminus\mathbb{D}_{\xi R}) \leq 1$ with $l(|z|) \equiv \frac{1}{\xi R}$. Therefore, for all $\xi \in (0, 1)$ and $R \in \left(0, \frac{672}{17}\right)$ inequality $N(\alpha, l) \leq 1$ holds with $l(|z|) \equiv \max\left\{\frac{1}{\xi R},\ \frac{672+11R}{(1-\xi)R(672-17R)}\right\}$.

If we put $\xi = \frac{672-17R}{1344-6R}$, then $\frac{1}{\xi R} = \frac{672+11R}{(1-\xi)R(672-17R)} = \frac{1344-6R}{R(672-17R)}$. Therefore for all $R \in (0,\ 672/17)$ we have $N(\alpha, l) \leq 1$ with $l(|z|) \equiv \frac{1344-6R}{R(672-17R)}$. The minimal value of the last function on $(0,\ 672/17)$ is $\frac{31+2\sqrt{238}}{336}$ if $R = 224\left(1-\sqrt{\frac{14}{17}}\right)$.

For the function $\beta(z)$ using recurrent formulas (8) we have

$$\sum_{k=1}^{\infty}|\beta_k|R^k = \frac{R}{80} + \sum_{k=2}^{\infty}|\beta_k|R^k = \frac{R}{80} + \sum_{k=2}^{\infty}\frac{|\beta_{k-1}|R^k}{16k(4k+1)} = \frac{R}{80} + \sum_{k=1}^{\infty}\frac{R}{16(k+1)(4k+5)}|\beta_k|R^k.$$

That is

$$\sum_{k=1}^{\infty}\left(1 - \frac{R}{16(k+1)(4k+5)}\right)|\beta_k|R^k = \frac{R}{80}. \tag{12}$$

Since $\frac{R}{16(k+1)(4k+5)} \leq \frac{R}{288}$, so if $R < 288$, then from (12) we get

$$\sum_{k=1}^{\infty}|\beta_k|R^k \leq \frac{R/80}{1-(R/288)} = \frac{18R}{1440-5R}.$$

Therefore to use lemma 2 it is necessary $R < \dfrac{1440}{23}$. For such R by lemma 2 we obtain $N(\beta, l;\, \mathbb{D}_{\xi R}) \le 1$ with $l(|z|) \equiv \dfrac{1440+13R}{(1-\xi)R(1440-23R)}$.

To investigate l-index boundedness of the function $\beta(z)$ in $\mathbb{C}\setminus\mathbb{D}_{\xi R}$ we use the fact that $\beta(z)$ is a solution of the equation (6), i.e. $\beta''(z) = -\dfrac{5}{4z}\beta'(z) + \dfrac{1}{64z}\beta(z)$. If $|z| \ge \xi R$, $R < 48$ and $l = 1/(\xi R)$ we have

$$\frac{|\beta''(z)|}{2!l^2} \le \frac{5}{8}\frac{|\beta'(z)|}{1!l} + \frac{\xi R}{128}|\beta(z)| \le \max\left\{\frac{|\beta'(z)|}{1!l},\, |\beta(z)|\right\}. \tag{13}$$

Let us differentiate the equation (6) n times. Then we obtain

$$64z\beta^{(n+2)}(z) + (64n+80)\beta^{(n+1)}(z) - \beta^{(n)}(z) = 0.$$

Therefore, if $|z| \ge \xi R$, $R < 48$ and $l = 1/(\xi R)$, then for all $n \in \mathbb{N}$ we get

$$\begin{aligned}\frac{|\beta^{(n+2)}(z)|}{(n+2)!l^{n+2}} &\le \frac{64n+80}{64(n+2)|z|l}\frac{|\beta^{(n+1)}(z)|}{(n+1)!l^{n+1}} + \frac{1}{64(n+2)(n+1)|z|l^2}\frac{|\beta^{(n)}(z)|}{n!l^n} \\ &\le \frac{64n+80}{64(n+2)}\frac{|\beta^{(n+1)}(z)|}{(n+1)!l^{n+1}} + \frac{\xi R/(n+1)}{64(n+2)}\frac{|\beta^{(n)}(z)|}{n!l^n} \le \max\left\{\frac{|\beta^{(n+1)}(z)|}{(n+1)!l^{n+1}}, \frac{|\beta^{(n)}(z)|}{n!l^n}\right\}.\end{aligned} \tag{14}$$

Inequalities (13) and (14) imply that for all $n \in \mathbb{Z}_+$ and $z \in \mathbb{C}\setminus\mathbb{D}_{\xi R}$ inequality

$$\frac{|\beta^{(n)}(z)|}{n!l^n} \le \max\left\{\frac{|\beta'(z)|}{1!l},\, |\beta(z)|\right\}$$

holds, that is $N(\beta, l;\, \mathbb{C}\setminus\mathbb{D}_{\xi R}) \le 1$ with $l(|z|) \equiv \dfrac{1}{\xi R}$. Therefore, for all $\xi \in (0,\, 1)$ and $R \in (0,\, 48)$ inequality $N(\beta, l) \le 1$ holds with $l(|z|) \equiv \max\left\{\dfrac{1}{\xi R}, \dfrac{1440+13R}{(1-\xi)R(1440-23R)}\right\}$.

If we put $\xi = \dfrac{1440-23R}{2880-10R}$, then $\dfrac{1}{\xi R} = \dfrac{1440+13R}{(1-\xi)R(1440-23R)} = \dfrac{2880-10R}{R(1440-23R)}$. Therefore, for all $R \in (0,\, 48)$ we have $N(\beta, l) \le 1$ with $l(|z|) \equiv \dfrac{2880-10R}{R(1440-23R)}$. The minimal value of the last function on $(0,\, 48)$ is $\dfrac{41+2\sqrt{414}}{720}$ if $R = 288\left(1-\sqrt{\dfrac{18}{23}}\right)$.

Therefore, the following proposition is true.

Proposition 1. $N(\alpha, l) \le 1$ *with* $l(|z|) \equiv \dfrac{31+2\sqrt{238}}{336}$ *and* $N(\beta, l) \le 1$ *with* $l(|z|) \equiv \dfrac{41+2\sqrt{414}}{720}$.

2 GEOMETRICAL PROPERTIES

We use following lemma to investigate convexity of the functions $\alpha(z)$ and $\beta(z)$.

Lemma 3 ([2]). *If* $\sum\limits_{n=2}^{+\infty} n^2|f_n| \le |f_1|$, *then function* (1) *is convex in* $\mathbb{D}$.

Using recurrent formula (7) we get

$$\sum_{n=2}^{+\infty} n^2|\alpha_n| \le 4|\alpha_2| + \sum_{n=3}^{+\infty} n^2 \frac{|\alpha_{n-1}|}{16n(4n-1)} = \frac{4}{10752} + \sum_{n=2}^{+\infty} \frac{n+1}{16n^2(4n+3)} n^2|\alpha_n|,$$

that is

$$\sum_{n=2}^{+\infty} \left(1 - \frac{n+1}{16n^2(4n+3)}\right) n^2|\alpha_n| \le \frac{1}{2688}. \tag{15}$$

Since for $n \ge 2$ we have the inequality $1 - \frac{n+1}{16n^2(4n+3)} \ge 1 - \frac{3}{704}$, so (15) implies

$$\sum_{n=2}^{+\infty} n^2|\alpha_n| \le \frac{1/2688}{701/704} < \frac{1}{48} = |\alpha_1|.$$

Applying a similar reasoning to the function $\beta(z)$ we obtain

$$\sum_{n=2}^{+\infty} n^2|\beta_n| \le 4|\beta_2| + \sum_{n=3}^{+\infty} n^2 \frac{|\beta_{n-1}|}{16n(4n+1)} = \frac{4}{23040} + \sum_{n=2}^{+\infty} \frac{n+1}{16n^2(4n+5)} n^2|\beta_n|,$$

that is

$$\sum_{n=2}^{+\infty} \left(1 - \frac{n+1}{16n^2(4n+5)}\right) n^2|\beta_n| \le \frac{1}{5760}.$$

Since $1 - \frac{n+1}{16n^2(4n+5)} \ge 1 - \frac{3}{832}$, so

$$\sum_{n=2}^{+\infty} n^2|\beta_n| \le \frac{1/5760}{829/832} \le \frac{1}{80} = |\beta_1|.$$

Therefore, the next proposition is true.

Proposition 2. *Functions $\alpha(z)$ and $\beta(z)$ are convex in $\mathbb{D}$.*

3 GROWTH

The next proposition describes the growth of the functions $\alpha(z)$ and $\beta(z)$.

Proposition 3. $\ln M_\alpha(r) = (1+o(1))\frac{\sqrt{r}}{4}$ *and* $\ln M_\beta(r) = (1+o(1))\frac{\sqrt{r}}{4}$ *as $r \to \infty$, where $M_f(r) = \max\{|f(z)| : |z| = r\}$.*

Really, since

$$\alpha_n = \frac{\alpha_{n-1}}{16n(4n-1)} = \frac{\alpha_0}{16^n n!} \prod_{k=1}^{n} \frac{1}{4k-1} = \frac{1}{64^n (n!)^2} \prod_{k=1}^{n} \left(1 + \frac{1}{4k-1}\right),$$

so for every $\varepsilon > 0$ and for all $n \in \mathbb{N}$

$$\frac{1}{64^n (n!)^2} \le \alpha_n \le \frac{K(1+\varepsilon)^n}{64^n (n!)^2}, \tag{16}$$

where $K = K(\varepsilon)$ is a positive constant.

To obtain an asymptotic behavior of the function $\alpha(z)$ from inequality (16) we will consider the function $g(r) = \sum\limits_{n=0}^{+\infty} \frac{r^n}{(n!)^2}$, where $r \geq 0$. Let $\mu_g(r) = \max\{r^n/(n!)^2 : n \geq 0\}$ be the maximal term of the last series and $\nu_g(r) = \max\{n : r^n/(n!)^2 = \mu_g(r)\}$ be the central index. Then $\nu_g(r) = n$ for $n^2 \leq r < (n+1)^2$, therefore $\nu_g(r) = (1+o(1))\sqrt{r}$ if $r \to +\infty$. Therefore

$$\ln \mu_g(r) = \ln \mu_g(1) + \int_1^r \frac{\nu_g(t)}{t} dt = (1+o(1))2\sqrt{r}, \quad r \to +\infty,$$

and by the Borel's theorem we get $\ln M_g(r) = (1+o(1)) \ln \mu_g(r) = (1+o(1))2\sqrt{r}$, $r \to +\infty$. From (16), in view of arbitrariness of ε, we have

$$\ln M_\alpha(r) = (1+o(1))2\sqrt{\frac{r}{64}} = (1+o(1))\frac{\sqrt{r}}{4}, \quad r \to +\infty.$$

Similar we get asymptotical equality $\ln M_\beta(r) = (1+o(1))\frac{\sqrt{r}}{4}$, $r \to +\infty$.

4 Main theorem

Propositions 1–3 imply the following theorem.

Theorem 1. *The general solution of* (4) *can be written in the form* $w(z) = C_1\alpha(z^4) + C_2 z\beta(z^4)$, *where entire functions* $\alpha(z)$ *and* $\beta(z)$ *are convex in* $\mathbb{D}$, $N(\alpha, l) \leq 1$ *with* $l(|z|) \equiv \frac{31+2\sqrt{238}}{336}$ *and* $N(\beta, l) \leq 1$ *with* $l(|z|) \equiv \frac{41+2\sqrt{414}}{720}$, *also* $\ln M_\alpha(r) = (1+o(1))\frac{\sqrt{r}}{4}$ *and* $\ln M_\beta(r) = (1+o(1))\frac{\sqrt{r}}{4}$ *as* $r \to \infty$, *where* $M_f(r) = \max\{|f(z)| : |z| = r\}$.

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Received 27.07.2015

Revised 23.10.2015

Трухан Ю.С. *Властивості розв'язків рівняння Вебера* // Карпатські матем. публ. — 2015. — Т.7, №2. — С. 247–253.

Досліджено властивості функцій $\alpha(z)$ та $\beta(z)$ таких, що $\alpha(z^4)$ та $z\beta(z^4)$ є лінійно незалежними розв'язками рівняння Вебера $w'' - (\frac{z^2}{4} - \nu - \frac{1}{2})w = 0$ при $\nu = -\frac{1}{2}$, а саме обмеженість l-індексу, опуклість та можливе зростання.

Ключові слова і фрази: ціла функція, обмеженість l-індексу, зростання, опукла функція, рівняння Вебера.